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On the method of alternating resolvents

Oganeditse A. Boikanyo *, Gheorghe Moroşanu

Department of Mathematics and its Applications, Central European University, Nador u. 9, H-1051 Budapest, Hungary

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To Professor Dorin Ieşan on the occasion of his 70th birthday

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ABSTRACT

The work of Hundal [H. Hundal, An alternating projection that does not converge in norm, Nonlinear Anal. 57 (1) (2004) 35–61] has revealed that the sequence generated by the method of alternating projections converges weakly, but not strongly in general. In this paper, we present several algorithms based on alternating resolvents of two maximal monotone operators, A and B, that can be used to approximate common zeros of A and B. In particular, we prove that the sequences generated by our algorithms converge strongly. A particular case of such algorithms enables one to approximate minimum values of certain convex functionals.

1. Introduction and preliminaries

Let H be a real Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. An operator $A : D(A) \subset H \to 2^H$ is called monotone if it satisfies the monotonicity property

$$\langle x - x', y - y' \rangle \geq 0, \quad \forall (x, y), (x', y') \in \text{graph}(A).$$

Equivalently, A is monotone if its graph is a monotone subset of the product space $H \times H$. If there is no monotone operator A' whose graph properly contains the graph of A, then A is called a maximal monotone operator. For a maximal monotone operator A, the resolvent of A, defined by $J_\beta^A := (I + \beta A)^{-1}$, is well defined on the whole space H and is single valued for every $\beta > 0$. Most importantly, J_β^A is nonexpansive; that is, for every $x, y \in H$, the inequality $\|J_\beta^A x - J_\beta^A y\| \leq \|x - y\|$ holds. See, for example, [1] for details.

We will use the following notations: given a sequence $(x_n)_{n \in \mathbb{N}_0}$, $\mathbb{N}_0 = \{0, 1, \ldots\}$, (or (x_n) in short), and a point $x \in H$, $x_n \to x$ (respectively, $x_n \rightharpoonup x$) means that (x_n) converges strongly (respectively, weakly) to x. The weak ω-limit set of (x_n) will be denoted by $\omega_w((x_n))$. This set is defined as follows:

$$\omega_w((x_n)) = \left\{ x \in H \mid x_{n_k} \rightharpoonup x \text{ for some subsequence } (x_{n_k})_{k \in \mathbb{N}_0} \text{ of } (x_n)_{n \in \mathbb{N}_0} \right\}.$$

The class of proper and convex functions from H into $(-\infty, \infty]$ will be denoted by $\Gamma(H)$. For any $\varphi \in \Gamma(H)$, the subdifferential (operator) $\partial\varphi : H \to H$ is defined by

$$\partial\varphi(x) = \{ w \in H \mid \varphi(x) - \varphi(v) \leq \langle w, x - v \rangle \text{ for all } v \in H \}.$$

* Corresponding author.
E-mail addresses: boikanyo_oganeditse@ceu-budapest.edu (O.A. Boikanyo), morosanug@ceu.hu (G. Moroşanu).

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A point $z \in H$ minimizes $\varphi \in \Gamma(H)$ iff $(z, 0) \in \partial\varphi$ (meaning that $z \in D(\partial\varphi)$ and $0 \in \partial\varphi(z)$). Recall that the subdifferential of a proper, lower semicontinuous and convex function is a maximal monotone operator. See, for example, [1, Theorem 1.12, p. 36] for details. Given a closed and convex subset C of H, the indicator function of C is a (proper) convex and lower semicontinuous function which gives the value zero at $x \in C$ and infinity outside C. Its subdifferential is the normal cone of C.

Now, let K_1 and K_2 be two nonempty, closed and convex sets in H with nonempty intersection, and consider the (convex feasibility) problem

$$\text{find an } x \in H \text{ such that } x \in K_1 \cap K_2. \tag{1}$$

The roots of this problem go as far back as the early 1930s, when von Neumann showed that, in the case when K_1 and K_2 are subspaces, the sequence of alternating projections

$$H \ni x_0 \mapsto x_1 = P_{K_1}x_0 \mapsto x_2 = P_{K_2}x_1 \mapsto x_3 = P_{K_1}x_2 \mapsto x_4 = P_{K_2}x_3 \mapsto \cdots$$

converges strongly to the point in the intersection of K_1 and K_2 which is the nearest to the starting point x_0. For the proof of this result, see, for example, [2] and the references therein. In 1965, Bregman [3] showed that, for two arbitrary closed and convex sets K_1 and K_2 with nonempty intersection, the sequence (x_n) generated by the method of alternating projections converges weakly to a point in $K_1 \cap K_2$. The question of whether or not (x_n) converges strongly remained open until recently, when Hundal [4] constructed an example in ℓ^2 showing that, for any starting point $x_0 \in \ell^2$, there exist a hyperplane K_1 and a cone K_2 such that $K_1 \cap K_2 = \{0\}$ and the sequence of alternating projections (x_n) converges weakly to zero, but not strongly; see also [5].

It should be noted that the projection operator coincides with the resolvent operator of a normal cone. Therefore, a natural way of extending the method of alternating projections is to consider two arbitrary maximal monotone operators, say A and B, instead of normal cones, in which case the method of alternating (or composition of) resolvents is defined as follows:

$$H \ni x_0 \mapsto x_1 = J^A_{\beta_1}x_0 \mapsto x_2 = J^B_{\gamma_1}x_1 \mapsto x_3 = J^A_{\beta_2}x_2 \mapsto x_4 = J^B_{\gamma_2}x_3 \mapsto \cdots,$$

where (β_n) and (γ_n) are sequences of positive real numbers. There are already many papers concerning this extension. In particular, for $\beta_n = \gamma_n = \lambda > 0$ for all $n \geq 1$, Bauschke et al. [6] showed that the sequence generated from this method converges weakly to a point of Fix $J^A_\lambda J^B_\lambda$ – the fixed point set of the composition $J^A_\lambda J^B_\lambda$ – provided that this set is not empty. In this paper, we investigate the convergence properties of the sequence generated by the method of alternating resolvents defined above for the case when (β_n) and (γ_n) are not constant sequences. More precisely, we consider the inexact iterative method

$$x_{2n+1} = J^A_{\beta_n}(x_{2n} + e_n) \quad \text{for } n = 0, 1, \ldots, \tag{2}$$

$$x_{2n} = J^B_{\gamma_n}(x_{2n-1} + e'_n) \quad \text{for } n = 1, 2, \ldots, \tag{3}$$

where $x_0 \in H$ is a given starting point. We will prove that, under a summability condition on $(\|e_n\|)$ and $(\|e'_n\|)$, where (e_n) and (e'_n) are sequences of computational errors, the sequence generated by (2) and (3) is weakly convergent to a point in $A^{-1}(0) \cap B^{-1}(0)$ provided that this set is not empty, and that both (β_n) and (γ_n) are bounded from below away from zero. In this connection, see also [7], where an inexact resolvent iterative method for a single monotone operator is considered. In order to obtain strong convergence results for general maximal monotone operators A and B, a modification (following the idea from the case of a single maximal monotone operator (see [8–12])) of this method is carried out; see Section 3. With such a modification, the summability condition on the error sequences (e_n) and (e'_n) is also relaxed.

2. Preliminary lemmas

In this section, we present the necessary tools needed to prove our main results. The first lemma, which is due to Xu [8], is basic, yet very useful.

Lemma 1 (*see [8]*)**.** *Let (s_n) be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - a_n)s_n + a_n b_n + c_n, \quad n \geq 0,$$

where (a_n), (b_n), and (c_n) satisfy the following conditions: (i) $(a_n) \subset [0, 1]$, *with* $\sum_{n=0}^{\infty} a_n = \infty$, (ii) $\limsup_{n\to\infty} b_n \leq 0$, *and* (iii) $c_n \geq 0$ *for all* $n \geq 0$ *with* $\sum_{n=0}^{\infty} c_n < \infty$. *Then* $\lim_{n\to\infty} s_n = 0$.

Weak convergence results are proved with the aid of the following known lemma due to Opial.

Lemma 2 (*Z. Opial, (see, e.g., [1, p. 5])*)**.** *Let F be a nonempty subset of H. Assume that the sequence (x_n) satisfies the following conditions:* (i) $\lim_{n\to\infty} \|x_n - q\| = \rho(q)$ *exists for all $q \in F$, and* (ii) *any weak cluster point of (x_n) belongs to F. Then, there exists a point $p \in F$ such that (x_n) converges weakly to p.*

Often, we shall use the following identity, the proof of which is well known and can easily be reproduced.