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Dissipative mechanism of a semilinear higher order parabolic equation in $\mathbb{R}^N$ ☆

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It is known that the concept of dissipativeness is fundamental for understanding the asymptotic behavior of solutions to evolutionary problems. In this paper we investigate the dissipative mechanism for some semilinear fourth-order parabolic equations in the spaces of Bessel potentials and discuss some weak conditions that lead to the existence of a compact global attractor. While for second-order reaction–diffusion equations the dissipativeness mechanism has already been satisfactorily understood (see Arrieta et al. (2004), doi:10.1142/S0218202504003234 [7]), for higher order problems in unbounded domains it has not yet been fully developed. As shown throughout the paper, one of the main differences from the case of reaction–diffusion equations stems from the lack of a maximum principle. Thus we have to rely here on suitable energy estimates for the solutions. As in the case of second-order reaction–diffusion equations, we show here that both linear and nonlinear terms have to collaborate in order to produce dissipativeness. Thus, the dissipative mechanisms in second-order and fourth-order equations are similar, although the lack of a maximum principle makes the proofs more difficult and the results not as complete.

Finally, we make essential use of the sharp results of Cholewa and Rodriguez-Bernal (2012), doi:10.1016/j.na.2011.08.022 [12], on linear fourth-order equations with a very large class of linear potentials.

1. Introduction

In this article we consider the following Cauchy problem in $\mathbb{R}^N$:

$$\begin{cases} u_t + \Delta^2 u = f(x,u), & t > 0,\ x \in \mathbb{R}^N, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where the nonlinear term is assumed to be of the general form

$$f(x,u) = g(x) + m(x)u + f_0(x,u), \quad x \in \mathbb{R}^N,\ u \in \mathbb{R}, \tag{1.2}$$

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for some suitable m, g described below and

$$f_0 : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R} \quad \text{is locally Lipschitz in } u \in \mathbb{R} \text{ uniformly for } x \in \mathbb{R}^N, \tag{1.3}$$

and

$$f_0(x,0) = 0, \qquad \frac{\partial f_0}{\partial u}(x,0) = 0, \quad x \in \mathbb{R}^N. \tag{1.4}$$

In some cases, depending on the space in which we solve (1.1), we will also require a growth condition in f_0 of the form

$$|f_0(x,u_1) - f_0(x,u_2)| \le c|u_1 - u_2|(1 + |u_1|^{\rho-1} + |u_2|^{\rho-1}), \quad u_1, u_2 \in \mathbb{R} \tag{1.5}$$

for some $\rho > 1$ and $c > 0$. Note that this class of nonlinear terms includes logistic type nonlinearities of the form

$$f(x,u) = g(x) + m(x)u - u|u|^{\rho-1}, \quad x \in \mathbb{R}^N,\ u \in \mathbb{R},\ \rho > 1$$

under mild assumptions on $g(x)$, $m(x)$.

Our goal here is to describe a general dissipative mechanism for this equation in a suitable functional setting. In this context "dissipative" refers to the properties that solutions of (1.1) are globally defined and bounded in several norms and moreover that they have a well defined asymptotic behavior.

In recent years the asymptotic behavior of solutions of evolutionary equations in unbounded domains has been studied by many authors and much progress has been achieved, especially for the case of reaction–diffusion problems. By this we mean that in (1.1), one replaces the term $\Delta^2 u$ by $-\Delta u$. See e.g., in chronological order, [1–8], where different conditions have been given to guarantee that the reaction–diffusion equation is dissipative and has a well defined asymptotic behavior in terms of a global attractor, a set which contains all the relevant asymptotic dynamics. Also, note that for the case of reaction–diffusion equations a very important tool is the maximum principle. This translates into the comparison principle for solutions of the reaction–diffusion problem or, in other words, into the monotonicity of the associated semigroup of solutions. Exploiting this tool it has been shown that, in great generality, when the problem is dissipative, it has extremal equilibria, which are the caps of the attractor; see [9–11].

Recall that in [7] a general mechanism was described for the dissipativeness of a reaction–diffusion problem in $\mathbb{R}^N$ and the existence of a global attractor. It was shown in this reference that the reaction and the diffusion have to collaborate to produce dissipativeness. If this does not happen, then the linear term in the equation is able to produce unbounded global solutions. This is in sharp contrast with the behavior of solutions in bounded domains where the dissipative character of the nonlinearity is enough, independently of the behavior of the linear term, to produce dissipativity. When this collaboration between reaction and diffusion occurs, then one finds suitable estimates for the solutions in $L^\infty(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ and furthermore, one finds that solutions remain small as $|x| \to \infty$ for large times. Both of these results become crucial for proving that the problem is dissipative. Also for both of them the arguments in [7] rely heavily on the comparison principle.

Hence, the primary goal of the present paper is to investigate the dissipative mechanism of the higher order parabolic equation in $\mathbb{R}^N$, (1.1). We remark that, since (1.1) involves a fourth-order elliptic operator in the main part, the maximum principle is no longer available. Thus, for the analysis of (1.1) we have to rely on some "energy" type estimates of solutions. This is why, although local existence for (1.1) can be obtained in more general spaces (see [12]), the asymptotic behavior of solutions is studied in an $L^2(\mathbb{R}^N)$ or in an $H^2(\mathbb{R}^N)$ setting.

In particular we show that, like for reaction–diffusion equations, if fourth-order diffusion and reaction collaborate, problem (1.1) is dissipative. This collaboration is reflected in the structure condition

$$uf(x,u) \le C(x)u^2 + D(x)|u|, \quad x \in \mathbb{R}^N,\ u \in \mathbb{R} \tag{1.6}$$

for some suitable functions $C(x)$ and $0 \le D(x)$ where $C(x)$ will be assumed to be such that the solutions of the linear problem

$$\begin{cases} u_t + \Delta^2 u = C(x)u, & t > 0,\ x \in \mathbb{R}^N, \\ u(0) = u_0 \in L^2(\mathbb{R}^N) \end{cases} \tag{1.7}$$

decay exponentially as $t \to \infty$.

More precisely, for (1.1), we will assume that in (1.6) we have

$$0 \le D \in L^s(\mathbb{R}^N), \qquad \max\left\{1, \frac{2N}{N+4}\right\} \le s \le 2, \quad (\text{and } s > 1 \text{ if } N = 4) \tag{1.8}$$

and

$$\sup_{y \in \mathbb{R}^N} \int_{B(y,1)} |C(x)|^r \, dx < \infty, \quad \text{for some } r > \max\left\{\frac{N}{4}, 1\right\}, \tag{1.9}$$

that is, $C \in L^r_U(\mathbb{R}^N)$, where this space is defined, for $1 \le r \le \infty$, as

$$L^r_U(\mathbb{R}^N) \stackrel{def}{=} \left\{\phi \in L^r_{loc}(\mathbb{R}^N) : \|\phi\|_{L^r_U(\mathbb{R}^N)} = \sup_{y \in \mathbb{R}^N} \|u\|_{L^r(B(y,1))} < \infty\right\}$$

(see [13,14] and note that $L^\infty_U(\mathbb{R}^N) := L^\infty(\mathbb{R}^N)$).