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Hopf-type formula defines viscosity solution for Hamilton–Jacobi equations with t-dependence Hamiltonian[☆]

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ABSTRACT

We prove that under some assumptions, the Hopf-type formula $u(t,x) = \max_{q\in\mathbb{R}^n}\{\langle x,q\rangle - \sigma^*(q) - \int_0^t H(\tau,q)d\tau\}$ defines a viscosity solution of the Cauchy problem for Hamilton–Jacobi equation (H,σ) where the initial condition σ is convex but the Hamiltonian $H = H(t,p)$ is not necessarily convex in p.

1. Introduction

Consider the Cauchy problem for Hamilton–Jacobi equations of the form

$$\frac{\partial u}{\partial t} + H(t,x,D_x u) = 0, \quad (t,x) \in \Omega = (0,T)\times\mathbb{R}^n, \tag{1.1}$$

$$u(0,x) = \sigma(x), \quad x\in\mathbb{R}^n. \tag{1.2}$$

The Hopf–Lax–Oleinik formula for solutions of the problem (1.1) and (1.2) is given by

$$u(t,x) = \min_{y\in\mathbb{R}^n}\left\{\sigma(y) + tH^*\left(\frac{x-y}{t}\right)\right\}, \tag{1.3}$$

where the Hamiltonian $H(t,x,p) = H(p)$ is convex and superlinear and σ is Lipschitz on $\mathbb{R}^n$.

If $H(t,x,p) = H(p)$ is a continuous function and $\sigma(x)$ is a convex Lipschitz function on $\mathbb{R}^n$, then the Hopf formula is

$$u(t,x) = \max_{q\in\mathbb{R}^n}\{\langle x,q\rangle - \sigma^*(q) - tH(q)\}. \tag{1.4}$$

Here H^* (resp., σ^*) denotes the Fenchel conjugate of the convex function H (resp., σ).

Under suitable assumptions, these formulas give the viscosity solutions to the above Cauchy problem [1]. If $H = H(t,x,p)$ is a convex function with respect to the last variable, then the general representation of the Hopf–Lax–Oleinik type formula (1.3) is established and studied thoroughly, using the calculus of variations; see [2] and references therein. If $H(t,x,\cdot)$ is

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nonconvex, the Hamiltonians $H = H(p)$ can be expanded to the forms $H = H(u, p)$ or $H = H(t, p)$ to get generalized solutions such as Lipschitz or viscosity solutions. Nevertheless, many questions remain open for the case of nonconvex Hamiltonians.

Naturally, it is expected that the formula

$$u(t,x) = \max_{q\in\mathbb{R}^n}\left\{\langle x, q\rangle - \sigma^*(q) - \int_0^t H(\tau, q)d\tau\right\} \tag{1.5}$$

defines a viscosity solution as a generalization of formula (1.4) under the assumptions that $H = H(t, p)$ is continuous and σ is convex.

Unfortunately, as in [3], Lions and Rochet showed that it is not a viscosity solution in general, although it is a Lipschitz solution under some compatible conditions between $H(t, p)$ and $\sigma(x)$; see [4]. The weakness of the notion of Lipschitz solution is that the uniqueness of the solution is not guaranteed.

Under different hypotheses, some results on the representation of the viscosity solution of the problem are established. For example, in [5] Silin constructed the notion of conjugate integral to deal the context where $H(t, p)$ satisfies the following condition

$$|H(t_1, p) - H(t_2, p)| \le \omega(|t_1 - t_2|),$$

for all $t_1,\ t_2 \in [0, T]$ and $p \in \mathbb{R}^n$. Here $\omega(\cdot)$ is a modulus.

This paper is devoted to proving that formula (1.5) defines a viscosity solution of the problem (1.1)–(1.2) for a class of Hamiltonians $H = H(t, p)$. In Section 2, we first recall some definitions and properties of semiconvex functions as well as sub- and superdifferential of a function. Then we prove that $u(t, x)$ defined by (1.5) is a semiconvex function. In Section 3, we check a *"consecutiveness property"* of $H = H(t, p)$ and $\sigma(x)$: it is the *"semigroup property"* in case $H = H(p)$ and $\sigma(x)$; see [3]. Furthermore, based on a property of the subdifferential of semiconvex functions, analogous to convex functions, we adapt the proof of Lions and Rochet in [3] to verify that $u(t, x)$ is actually a viscosity solution. Next, we use the relationship between subdifferential $D^-u(t, x)$ and the set of all reachable gradients $D^*u(t, x)$ to prove that $u(t, x)$ is also a viscosity solution in case Hamiltonian $H(t, p)$ is concave in p.

The results obtained here are new for a large class of nonconvex Hamiltonians $H(t, p)$. However, they may not exhaust all possible cases for $u(t, x)$ to be a viscosity solution.

We use the following notations. Let T be a positive number, $\Omega = (0, T) \times \mathbb{R}^n$; $|\,.\,|$ and $\langle ., .\rangle$ be the Euclidean norm and the scalar product in $\mathbb{R}^n$, respectively, and let $B'(x_0, r)$ be the closed ball centered at x_0 with radius r.

2. Preliminaries and semiconvexity of the Hopf-type formula

2.1. Assumptions

We now consider the Cauchy problem for the Hamilton–Jacobi equation:

$$\frac{\partial u}{\partial t} + H(t, D_x u) = 0, \quad (t, x) \in \Omega = (0, T) \times \mathbb{R}^n, \tag{2.1}$$

$$u(0, x) = \sigma(x), \quad x \in \mathbb{R}^n, \tag{2.2}$$

where the Hamiltonian $H(t, p)$ is of class $C([0, T] \times \mathbb{R}^n)$ and $\sigma(x) \in C(\mathbb{R}^n)$ is a convex function.

Let σ^* be the Fenchel conjugate of σ. We denote by

$$D = \operatorname{dom} \sigma^* = \{y \in \mathbb{R}^n \mid \sigma^*(y) < +\infty\}$$

the effective domain of the convex function σ^*.

Throughout the paper, we use the following assumptions:

(A1): For every $(t_0, x_0) \in [0, T) \times \mathbb{R}^n$, there exist positive constants r and N such that

$$\langle x, p\rangle - \sigma^*(p) - \int_0^t H(\tau, p)d\tau < \max_{|q|\le N}\left\{\langle x, q\rangle - \sigma^*(q) - \int_0^t H(\tau, q)d\tau\right\},$$

whenever $(t, x) \in [0, T) \times \mathbb{R}^n,\ |t - t_0| + |x - x_0| < r$ and $|p| > N$.

(A2): $H(t, p)$ is admitted as one of two following forms:

(a) $H(t, \cdot)$ is a convex function for all $t \in (0, T)$.
(b) $H(t, p) = g(t)h(p) + k(t)$ for some functions $g,\ h,\ k$ where $g(t)$ does not change its sign for all $t \in (0, T)$.

Remark 2.1. (a) The assumption (A1) can be considered as a compatible condition for the Hamiltonian $H(t, p)$ and initial data $\sigma(x)$, primarily used in [6].
(b) If $\sigma(x)$ is Lipschitz continuous on $\mathbb{R}^n$, then $D = \operatorname{dom}\sigma^*$ is bounded. Consequently, the condition (A1) is automatically satisfied.
(c) If $H(t, p) = H(p)$, then the condition (A2), (b) is obviously satisfied.