Nonlinear Analysis 75 (2012) 3641–3671

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Strong polyhedral approximation of simple jump sets

Tuomo Valkonen

Institute for Mathematics and Scientific Computing, Karl-Franzens University of Graz, Heinrichstraße 36, A-8010 Graz, Austria

ARTICLE INFO

Article history:
Received 17 October 2011
Accepted 20 January 2012
Communicated by S. Ahmad

MSC:
26B30
49Q20
74R10

Keywords:
Bounded variation
Approximation
Discontinuities

ABSTRACT

We prove a strong approximation result for functions $u \in W^{1,\infty}(\Omega \setminus J)$, where J is the union of finitely many Lipschitz graphs satisfying some further technical assumptions. We approximate J by a polyhedral set in such a manner that a regularisation term $\eta(\mathrm{Div}^j u^i)$, $(i = 0, 1, 2, \ldots)$, is convergent. The boundedness of this regularisation functional itself, introduced in [T. Valkonen, Transport equation and image interpolation with SBD velocity fields, J. Math. Pures Appl. 95 (2011) 459–494. doi:10.1016/j.matpur.2010.10.010] ensures the convergence in total variation of the jump part $\mathrm{Div}^j u^i$ of the distributional divergence.

1. Introduction

Let $u \in \mathrm{SBV}(\Omega)$ be a special function of bounded variation on the domain $\Omega \subset \mathbb{R}^m$. We would like to approximate u by a sequence of functions $\{u^i\}_{i=0}^{\infty}$ such that u^i is reasonably smooth in $\Omega \setminus \widehat{J}_{u^i}$, $(i = 0, 1, 2, \ldots)$, and $\widehat{J}_{u^i}$ is a polyhedral $(m-1)$-dimensional set, containing the jump set J_{u^i}. As the novelty of our results, we would like convergence from a regularisation term $\eta(\mathrm{Div}^j u^i)$, introduced in [1]. The boundedness of this term ensures that if $\mathrm{Div}^j u^i \overset{*}{\rightharpoonup} \mathrm{Div}^j u$ and $|\mathrm{Div}^j u^i| \overset{*}{\rightharpoonup} \lambda$, then $\lambda = |\mathrm{Div}^j u|$. The notation $\mathrm{Div}^j u$ here stands for the "jump part" of the distributional divergence $\mathrm{Div}\, u$, while the absolutely continuous part will be denoted by $\mathrm{div}\, u$.

Why do we want this kind of strong approximation property? In [1] we studied an extension of the transport equation involving "jump sources and sinks". With $u = (1, b)$ the velocity field and I the space–time data being transported, it can be stated as

$$\mathrm{Div}(Iu) - I\,\mathrm{div}\, u - \tau\,\mathrm{Div}^j u = 0 \tag{1}$$

for some τ defined on the jump set of u, modelling the sources and sinks. To show the stability of (1) with $\{I^i\}_{i=0}^{\infty}$ converging weakly in $\mathrm{BV}(\Omega)$ and $\{u^i\}_{i=0}^{\infty}$ converging as in the SBV/SBD compactness theorems [2,3], we needed to further assume that $|\mathrm{Div}^j u^i|(\Omega) \to |\mathrm{Div}^j u|(\Omega)$. To use (1) as a constraint in an optimisation problem (specifically, image interpolation), we thus had to introduce the regularisation term $\eta(\mathrm{Div}^j u^i)$ ensuring this convergence. One possibility for the definition is

$$\eta(\mu) := \sum_{\ell=0}^{\infty} \left(|\mu|(\Omega) - 2^{-\ell m} \int_{\mathbb{R}^m} |\mu(x + [0, 2^{-\ell}]^m)|\, dx \right), \quad (\mu \in \mathcal{M}(\Omega)). \tag{2}$$

Roughly $\eta(\mu) < \infty$ says that on average the differences $2^{-\ell m}(|\mu|(x + [0, 2^{-\ell}]^m) - |\mu(x + [0, 2^{-\ell}]^m)|)$ go to zero as the scale $2^{-\ell}$ becomes smaller. Thus on small sets $|\mu|$ is close to μ.

E-mail address: tuomo.valkonen@iki.fi.

0362-546X/$ – see front matter © 2012 Elsevier Ltd. All rights reserved.
doi:10.1016/j.na.2012.01.022

The problem then becomes: can we, at least in principle, numerically solve problems involving such regularisation terms? That is, can we in particular construct a sequence of discretizations of u such that $\eta(\mathrm{Div}^j u^i) \to \eta(\mathrm{Div}^j u)$ along with the standard convergences $u^i \to u$ and $\nabla u^i \to \nabla u$ in L^2, $D^j u^i \to D^j u$ weakly*, and $\mathcal{H}^{m-1}(J_{u^i}) \to \mathcal{H}^{m-1}(J_u)$? In the present work, we intend to provide a partial answer. Specifically, we restrict our attention to functions $u \in W^{1,\infty}(\Omega \setminus \widehat{J_u})$, where $\widehat{J_u}$ is the union of finitely many Lipschitz graphs with bounded variation gradient mapping, satisfying further technical conditions, given in Definition 8. Assuming these conditions, we show that u can be approximated by functions $u^i \in W^{1,\infty}(\Omega \setminus \widehat{J_{u^i}})$ with $\widehat{J_{u^i}}$ polyhedral and satisfying Definition 8. Some of our proof techniques resemble those of the SBD approximation theorem of Chambolle [4,5]. In SBV a counterpart approximation theorem is proved by quite different techniques by Cortesani and Toader [6]. Their result provides largely similar convergence properties as ours, but is missing the crucial convergence of $\eta(\mathrm{Div}^j u^i)$. Of course, the class of functions that we are able to study at the moment is significantly smaller. Finally, we also study anisotropic approximation with $\widehat{J_{u^i}}$ restricted to lie on translations of the coordinate planes.

We have organised this paper as follows. First, in Section 2, we introduce notation and some other well-known tools. In Section 3 we study the functional η, and estimates for bounding it. As a consequence we also obtain some new SBV compactness results. In Section 4 we provide a series of further technical lemmas of general nature, needed to prove the approximation theorem. In the subsequent Section 5 we then introduce in detail the space where the approximated function u lies in, and provide further technical lemmas regarding the covering of the boundary of the jump set by cubes. Our main approximation theorem is then stated and proved in Section 6. Finally, we study anisotropic approximation in Section 7.

2. Preliminaries

2.1. Sets and functions

We denote the unit sphere in $\mathbb{R}^m$ by S^{m-1}, while the open ball of radius ρ centred at $x \in \mathbb{R}^m$ we denote by $B(x, \rho)$. The boundary of a set A is denoted ∂A, and the closure by $\mathrm{cl}\, A$.

For $\nu \in \mathbb{R}^m$, the hyperplane orthogonal to ν we denote by $\nu^\perp := \{z \in \mathbb{R}^m \mid \langle \nu, z\rangle = 0\}$. P_ν denotes the projection onto the subspace spanned by ν, and $P_\nu^\perp$ the projection onto $\nu^\perp$.

We denote by $\{e_1, \ldots, e_m\}$ the standard basis of $\mathbb{R}^m$.

The k-dimensional Jacobian of a linear map $L : \mathbb{R}^k \to \mathbb{R}^m$, $(k \le m)$, is defined as $\mathcal{J}_k[L] := \sqrt{\det(L^* \circ L)}$.

A set $\Gamma \subset \mathbb{R}^m$ is a called a Lipschitz d-graph (of Lipschitz factor L), if there exist a unit vector z_Γ, an open set V_Γ on a d-dimensional subspace of $z_\Gamma^\perp$, and a Lipschitz map $g_\Gamma : V_\Gamma \to \mathbb{R}^m$ of Lipschitz factor at most L, such that

$$\Gamma = \{y \in \mathbb{R}^m \mid g_\Gamma(v) = y, v = P_{z_\Gamma}^\perp y \in V_\Gamma\}.$$

We say that Γ is *polyhedral* if g_Γ is piecewise affine and V_Γ is a polyhedral set, i.e., consists of finitely many simplices. If g_Γ is further affine, we say that Γ is *affine*. We define the boundary as $\partial\Gamma := g_\Gamma(\partial V_\Gamma)$.

Remark 1. Consider the situation $d = m - 1$. If Γ is the graph of $f : U \subset \mathbb{R}^{m-1} \to \mathbb{R}$, then $g_\Gamma(v) = (x, f(x))$ for $v = (x, 0) \in V_\Gamma = U \times \{0\}$. More generally, if $V_\Gamma \subset z_\Gamma^\perp$ for some $z_\Gamma \in \mathbb{R}^m$, and $f : V_\Gamma \to \mathbb{R}$ is Lipschitz map, then $g_\Gamma(v) = v + z_\Gamma f(v)$ defines a Lipschitz graph. Conversely, if Γ is a Lipschitz graph per the above definition, then defining $f_\Gamma(v) := \langle g_\Gamma(v), z_\Gamma\rangle$ for $v \in V_\Gamma$, we obtain the more conventional description

$$\Gamma = \{v + f_\Gamma(v) z_\Gamma \mid v \in V_\Gamma\}.$$

For our purposes it is more convenient to work with the map g_Γ, however.

2.2. Measures

The space of (signed) Radon measures on an open set Ω is denoted $\mathcal{M}(\Omega)$. If V is a vector space, then the space of V-valued Radon measures on Ω is denoted $\mathcal{M}(\Omega; V)$. The k-dimensional Hausdorff measure, on any given ambient space $\mathbb{R}^m$, $(k \le m)$, is denoted by $\mathcal{H}^k$, while $\mathcal{L}^m$ denotes the Lebesgue measure on $\mathbb{R}^m$. For a measure μ and a measurable set A, we denote by $\mu \llcorner A$ the restriction measure defined by $(\mu \llcorner A)(B) := \mu(A \cap B)$. The total variation measure of μ is denoted $|\mu|$. For a Borel map $u : \Omega \to \mathbb{R}$ we denote $\mu(u) := \int_\Omega u\, d\mu$.

A measure $\mu \in \mathcal{M}(\Omega)$ is said to be *Ahlfors-regular* (in dimension d), if there exists $M \in (0, \infty)$ such that

$$M^{-1} r^d \le |\mu|(B(x, r)) \le M r^d \quad \text{for all } r > 0 \text{ and } x \in \mathrm{supp}\, \mu.$$

If only the first or the second inequality holds, then μ is said to be, respectively, *lower* or *upper* Ahlfors-regular.

We will often refer to the following standard result on weak* convergence. (See, e.g., [7, Proposition 1.62].)

Proposition 1. *Let $\mu^i \in \mathcal{M}(\Omega)$, $(i = 0, 1, 2, \ldots)$, be such that $\mu^i \stackrel{*}{\rightharpoonup} \mu \in \mathcal{M}(\Omega)$, and $|\mu^i| \stackrel{*}{\rightharpoonup} \lambda \in \mathcal{M}(\Omega)$. If E is a relatively compact μ-measurable set such that $\lambda(\partial E) = 0$, then $\mu^i(E) \to \mu(E)$. More generally, let $u : \Omega \to \mathbb{R}$ be any compactly supported Borel function, and denote by E_f the set of its discontinuity points. Then, if $\lambda(E_f) = 0$, we have $\int_\Omega u\, d\mu^i \to \int_\Omega u\, d\mu$.*