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Blow-up of rough solutions to the fourth-order nonlinear Schrödinger equation ☆

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ABSTRACT

This paper deals with the formation of singularities of rough blow-up solutions to the fourth-order nonlinear Schrödinger equation. The limiting profile and L^2-concentration of the rough blow-up solutions are obtained in $H^s(\mathbb{R}^4)$ with $s > s_0$, where $s_0 \leq \frac{9+\sqrt{721}}{20} \approx 1.793$. The new ingredient relies on the refined compactness result developed by Zhu et al. [S.H. Zhu, J. Zhang, H. Yang, Limiting profile of the blow-up solutions for the fourth-order nonlinear Schrödinger equation, Dyn. Partial Differ. Equ. 7 (2010) 187–205].

1. Introduction

In this paper, we study the Cauchy problem of the following fourth-order nonlinear Schrödinger equation

$$\mathrm{i}u_t - \Delta^2 u + |u|^2 u = 0, \quad t \geq 0,\ x \in \mathbb{R}^4, \tag{1.1}$$

$$u(0,x) = u_0, \tag{1.2}$$

where $\mathrm{i} = \sqrt{-1}$; $\Delta^2 = \Delta\Delta$ is the biharmonic operator defined in $\mathbb{R}^4$ and $\Delta = \sum_{j=1}^4 \frac{\partial^2}{\partial x_j^2}$ is the Laplace operator in $\mathbb{R}^4$; $u = u(t,x)$: $[0,T^*) \times \mathbb{R}^4 \to \mathbb{C}$ is the complex valued function and $0 < T^* \leq +\infty$. Fourth-order Schrödinger equations are introduced by Karpman [1], Karpman and Shagalov [2] to take into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity, and such fourth-order Schrödinger equations are written as

$$\mathrm{i}\phi_t + \varepsilon\Delta^2\phi + \mu\Delta\phi + |\phi|^{p-1}\phi = 0, \qquad \phi = \phi(t,x) : I \times \mathbb{R}^d \to \mathbb{C}, \tag{1.3}$$

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where ε and μ are two parameters, d is the space dimension. Note that Eq. (1.1) is a special case of Eq. (1.3) by taking $\varepsilon = -1$, $\mu = 0$ and $p = 1 + \frac{8}{d} = 3$. Eq. (1.1) is called the mass-critical due to the mass $M(u) = \int_{\mathbb{R}^4} |u(t,x)|^2 \mathrm{d}x$ and the equation itself are invariant under the rescaling symmetry $u \mapsto \lambda^2 u(\lambda^4 t, \lambda x)$.

We recall some known results for the classical focusing mass-critical nonlinear Schrödinger equation

$$\mathrm{i}v_t + \Delta v + |v|^{\frac{4}{d}} v = 0, \qquad v(0,x) = \varphi, \tag{1.4}$$

where $v = v(t,x) : I \times \mathbb{R}^d \to \mathbb{C}$. Ginibre and Velo [3] showed the local well-posedness in $H^1(\mathbb{R}^d)$. In this space energy arguments apply, and a blow-up theory has been developed in the last two decades (see [4–6] and the references therein). On the other hand, Cazenave [4] established the local well posedness in $H^s(\mathbb{R}^d)$ with $0 \le s < 1$. In this space, the energy arguments fail and studying the rough blow-up solutions is more difficult and interesting. Recently, many researchers are attracted to study the qualitative properties of blow-up solutions in lower regular space $H^s(\mathbb{R}^d)$ with $0 \le s < 1$ (see [7–12]). Combining the harmonic techniques with variational characteristic of the ground state, Colliander et al. [8] firstly obtained the mass concentration properties of the radially symmetric blow-up solutions in $H^s(\mathbb{R}^2)$ with $1 > s > \frac{1+\sqrt{11}}{5}$. Hmidi and Keraani [10] extended this result to the general blow-up solutions. Tzirakis [13] obtained the analogue results in $H^s(\mathbb{R}^1)$ with $1 > s > \frac{10}{11}$. Visan and Zhang [12] extended Colliander et al.'s results [8] to the general blow-up solutions in dimension $d \ge 3$ for some $1 > s > s_0$. For $\varphi \in L^2(\mathbb{R}^2)$, Bourgain [14] showed that some small amount of mass must concentrate in parabolic windows (at least along a subsequence). Keraani [15], Bégout and Vargas [16], Chae et al. [17] extended Bourgain's results [14] to $d = 1$ and $d \ge 3$.

In Eq. (1.1), if one replaces the nonlinearity $|u|^2u$ with $|u|^{p-1}u$, it is a class of semilinear fourth-order Schrödinger equations similar to Eq. (1.1), which has been widely investigated. For $1 < p < \frac{2d}{(d-4)^+}$ (we use the convention: $\frac{2d}{(d-4)^+} = +\infty$ when $d \le 4$ and $\frac{2d}{(d-4)^+} = \frac{2d}{d-4}$ when $d > 4$), Ben-Artzi et al. [18] established the local well-posedness in $H^2(\mathbb{R}^d)$. Fibich et al. [19] obtained the general results of global well-posedness in $H^2(\mathbb{R}^d)$. Pausader [20] and Segata [21] studied the global well-posedness and scattering of the fourth-order nonlinear Schrödinger equation with cubic nonlinearity. For $p = \frac{2d}{d-4}$, Miao et al. [22], Pausader [23] studied the global existence and scattering of the focusing fourth-order nonlinear Schrödinger equation; Miao et al. [24], Pausader [25] studied the global existence and scattering of the defocusing fourth-order nonlinear Schrödinger equation. The above studies focused on global solutions.

In the present paper, we study the limiting profile and L^2-concentration of blow-up solutions to the Cauchy problem (1.1)–(1.2) in $H^s := H^s(\mathbb{R}^4)$ with $0 < s < 2$. We first recall some known results for the Cauchy problem (1.1)–(1.2) in $H^2 := H^2(\mathbb{R}^4)$. Kenig et al. [26], Ben-Artzi et al. [18], Pausader [25] established the local well-posedness in H^2. Moreover, for the solution $u(t,x)$ of the Cauchy problem (1.1)–(1.2), there are two conservation laws in H^2:

(i) Conservation of mass

$$\int_{\mathbb{R}^4} |u(t,x)|^2 \mathrm{d}x = \int_{\mathbb{R}^4} |u_0|^2 \mathrm{d}x. \tag{1.5}$$

(ii) Conservation of energy

$$E(u(t)) := \frac{1}{2}\int_{\mathbb{R}^4} |\Delta u(t,x)|^2 \mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^4} |u(t,x)|^4 \mathrm{d}x = E(u_0). \tag{1.6}$$

On the other hand, we mean a special periodic solution of Eq. (1.1) in the form $u(t,x) = Q(x)\mathrm{e}^{-\mathrm{i}t}$ by the standing wave. It is easy to check that $Q(x)$ satisfies

$$\Delta^2 Q + Q - |Q|^2 Q = 0, \quad Q \in H^2, \tag{1.7}$$

and Q is called the ground state of Eq. (1.7) (see [19]). Zhu et al. [27] showed the existence of the ground state of Eq. (1.7). Levandosky [28] discussed the stability of the standing waves. Fibich et al. [19] showed some numerical observations of the solution to the Cauchy problem (1.1)–(1.2), which implies that if the initial data $\|u_0\|_{L^2} < \|Q\|_{L^2}$, then the solution $u(t,x)$ exists globally; if the initial data $\|u_0\|_{L^2} \ge \|Q\|_{L^2}$, then the solution $u(t,x)$ may blow up in finite time. Because of the effect of fourth-order dispersion $\Delta^2 u$, whether the variance identity arguments can be extended to show the existence of blow-up solutions for the fourth-order nonlinear Schrödinger equation is still unknown (see [29,19]). However, the numerical observations in [19] showed the existence of blow-up solutions. Baruch et al. [29], Zhu et al. [27] showed some dynamical properties of the blow-up solutions in the energy space H^2, such as blow-up rate, L^2-concentration and limiting profile. Meanwhile, Zhu et al. [27] obtained a refined compactness result, which is of particular significance to study the dynamical properties of blow-up solutions to fourth-order nonlinear Schrödinger equations.

Theorem 1.1. *Let $d = 4$ and $\{v_n\}_{n=1}^{+\infty}$ be a bounded sequence in H^2 such that*

$$\lim_{n\to+\infty} \sup \|\Delta v_n\|_{L^2} \le M \quad \text{and} \quad \lim_{n\to+\infty} \sup \|v_n\|_{L^4} \ge m. \tag{1.8}$$