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Lipschitz-like property of an implicit multifunction and its applications☆

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ABSTRACT

The aim of this work is twofold. First, we use the advanced tools of modern variational analysis and generalized differentiation to study the Lipschitz-like property of an implicit multifunction. More explicitly, new sufficient conditions in terms of the Fréchet coderivative and the normal/Mordukhovich coderivative of parametric multifunctions for this implicit multifunction to have the Lipschitz-like property at a given point are established. Then we derive sufficient conditions ensuring the Lipschitz-like property of an efficient solution map in parametric vector optimization problems by employing the above implicit multifunction results.

1. Introduction

The paper mainly deals with the stability theory of implicit multifunctions and parametric vector optimization problems. We first give some notation and definitions.

Let X, Y be Banach spaces and (P, d) be a metric space, and let $F : P \times X \rightrightarrows Y$ be a parametric multifunction. By means of this parametric multifunction one can define an *implicit multifunction* $G : P \rightrightarrows X$ as follows:

$$G(p) := \{x \in X \mid 0 \in F(p, x)\}. \tag{1.1}$$

Let $K \subset Y$ be a pointed, closed and convex cone with an apex at the origin.

Definition 1.1. We say that $y \in A$ is an *efficient point* of a subset $A \subset Y$ with respect to K if and only if $(y - K) \cap A = \{y\}$. The set of efficient points of A is denoted by $\mathrm{Eff}_K A$. We stipulate that $\mathrm{Eff}_K \emptyset = \emptyset$.

Given a vector function $f : P \times X \to Y$, we consider the following *parametric vector optimization problem*:

$$\mathrm{Eff}_K \left\{ f(p, x) \mid x \in X \right\}, \tag{1.2}$$

where x is the *unknown* (decision variable) and $p \in P$ a *parameter*.

For each $p \in P$, we put

$$\mathcal{F}(p) := \mathrm{Eff}_K \{ f(p, x) \mid x \in X \} \tag{1.3}$$

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and call $\mathcal{F} : P \rightrightarrows Y$ the *efficient point multifunction* of (1.2). One writes $x \in \mathcal{S}(p)$ to indicate that $x \in X$ is an *efficient solution* of (1.2) if $f(p, x) \in \mathcal{F}(p)$. The multifunction $\mathcal{S} : P \rightrightarrows X$ assigns to p the set of all efficient solutions of (1.2), i.e.,

$$\mathcal{S}(p) := \left\{x \in X \mid f(p, x) \in \mathcal{F}(p)\right\}, \tag{1.4}$$

is called the *efficient solution map* of (1.2).

Stability analysis in implicit multifunctions/or vector optimization problems has been investigated intensively by many researchers. One of the main problems here is to find sufficient conditions for the implicit multifunction G/or the efficient solution map $\mathcal{S}$ to have a certain stability property such as lower (upper) semi-continuous, continuous properties, calmness, Aubin (Lipschitz-like, pseudo-Lipschitz) properties, Lipschitz properties, and Hölder continuities. We refer the reader to [1–17] and their references therein for more details and discussions.

In this paper we first use the advanced tools of modern variational analysis and generalized differentiation to study the Lipschitz-like property of the implicit multifunction G defined in (1.1). More precisely, new sufficient conditions in terms of the Fréchet coderivative and the normal/Mordukhovich coderivative [18] of parametric multifunctions for this implicit multifunction G to be Lipschitz-like at a given point are established. These results generalize some corresponding results in [10,12]. Then we derive sufficient conditions ensuring the Lipschitz-like property of an efficient solution map $\mathcal{S}$ defined in (1.4) in parametric vector optimization problems by exploiting the above implicit multifunction results. The main tools for the proofs of our main results involve the *Ekeland variational principle* [19], the *nonsmooth version of Fermat's rule* (see e.g., [18]), the *fuzzy sum rule* for the Fréchet subdifferential (see e.g., [18]).

The rest of the paper is organized as follows. In Section 2, we first provide the basic definitions and notation from variational analysis and set-valued analysis. Then we recall some known auxiliary results which will be useful hereafter. Section 3 is devoted to providing sufficient conditions for the implicit multifunction G to be Lipschitz-like at a given point. In the last section we derive the Lipschitz-like property of the efficient solution map $\mathcal{S}$ of (1.2) by means of exploiting the implicit multifunction results given in Section 3.

2. Preliminaries and auxiliary results

Throughout the paper we use the standard notation of variational analysis and generalized differentiation; see, e.g., [18,20,21]. Unless otherwise stated, all spaces under consideration are Banach spaces whose norms are always denoted by $\|\cdot\|$. The canonical pairing between X and its topological dual X^* is denoted by $\langle\cdot,\cdot\rangle$. In this setting, w^* denotes the weak* topology in X^*, and A^* denotes the adjoint operator of a linear continuous operator A. The symbols B_X and B_{X^*} stand for the closed unit balls of X and its topological dual X^*, respectively. The closed ball with center x and radius ρ is denoted by $B_\rho(x)$. As usual, the distance from $u \in X$ to $\Omega \subset X$ is denoted by $\operatorname{dist}(u, \Omega) := \inf_{x \in \Omega} \|x - u\|$.

Given a set-valued mapping $F : X \rightrightarrows X^*$ between a Banach space X and its topological dual X^*, we denote by

$$\operatorname*{Lim\,sup}_{x \to \bar{x}} F(x) := \left\{x^* \in X^* | \exists \text{ sequences } x_k \to \bar{x} \text{ and } x_k^* \xrightarrow{w^*} x^* \text{ with } x_k^* \in F(x_k) \text{ for all } k \in \mathbb{N}\right\}$$

the *sequential Painlevé–Kuratowski upper/outer limit* with respect to the norm topology of X and the weak* topology of X^*, where $\mathbb{N} := \{1, 2, \ldots\}$.

Given $\Omega \subset X$ and $\varepsilon \geq 0$, define the collection of *ε-normals* to Ω at $\bar{x} \in \Omega$ by

$$\widehat{N}_\varepsilon(\bar{x}; \Omega) := \left\{x^* \in X^* | \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \leq \varepsilon\right\}, \tag{2.1}$$

where $x \xrightarrow{\Omega} \bar{x}$ means that $x \to \bar{x}$ with $x \in \Omega$. When $\varepsilon = 0$, the set $\widehat{N}(\bar{x}; \Omega) := \widehat{N}_0(\bar{x}; \Omega)$ in (2.1) is a cone called the *prenormal cone* or the *Fréchet normal cone* to Ω at $\bar{x}$.

The *Mordukhovich normal cone* $N(\bar{x}; \Omega)$ is obtained from $\widehat{N}_\varepsilon(x; \Omega)$ by taking the sequential Painlevé–Kuratowski upper limit in the weak* topology of X^* as

$$N(\bar{x}; \Omega) := \operatorname*{Lim\,sup}_{\substack{x \xrightarrow{\Omega} \bar{x} \\ \varepsilon \downarrow 0}} \widehat{N}_\varepsilon(x; \Omega), \tag{2.2}$$

where one can put $\varepsilon = 0$ when Ω is closed around $\bar{x}$ and the space X is *Asplund*, i.e., a Banach space whose separable subspaces have separable duals. Here, Ω is said to be (locally) *closed* around $\bar{x}$ if there is a neighborhood U of $\bar{x}$ such that $\Omega \cap U$ is closed. It is well known that the class of Asplund spaces is sufficiently large containing, in particular, all reflexive spaces and all spaces with separable duals. We refer the reader who is interested in this class, to the books [22,18,20] for numerous its characterizations and discussions on its applications.

Let $F : X \rightrightarrows Y$ be a set-valued mapping between Banach spaces with the domain and the graph

$$\operatorname{dom} F := \{x \in X \mid F(x) \neq \emptyset\}, \quad \operatorname{gph} F := \left\{(x, y) \in X \times Y | y \in F(x)\right\}.$$