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Optimal control problem for viscous Cahn–Hilliard equation

Xiaopeng Zhao *, Changchun Liu

Department of Mathematics, Jilin University, Changchun 130012, PR China

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ABSTRACT

This paper is concerned with the viscous Cahn–Hilliard equation, which arises in the dynamics of viscous first order phase transitions in cooling binary solutions. The optimal control under boundary condition is given and the existence of optimal solution to the equation is proved.

1. Introduction

Let $\Omega\subset\mathbb{R}$ be a bounded domain. Consider the viscous Cahn–Hilliard equation

$$u_t+\gamma u_{xxxx}-ku_{xxt}=\varphi(u)_{xx},\quad x\in\Omega, \tag{1}$$

with the boundary conditions

$$u(x,t)=u_{xx}(x,t)=0,\quad x\in\partial\Omega, \tag{2}$$

and the initial condition

$$u(x,0)=u_0(x),\quad x\in\Omega, \tag{3}$$

where $\gamma>0$, $k\geq 0$, $\varphi(u)=u^3-u$.

Eq. (1) arises from modeling the dynamics of viscous first order phase transitions in cooling binary solutions such as alloys, glasses and polymer mixtures, see for example [1–3]. In (1), u is the concentration of one of the two phases, k represents the viscosity coefficient, γ is the interfacial energy parameter, $\varphi(u)$ is the intrinsic chemical potential. If $k=0$, we obtain the standard Cahn–Hilliard equation.

During the past years, only a few papers were devoted to the viscous Cahn–Hilliard equation. In [4], Liu and Yin considered the viscous Cahn–Hilliard equation (1) for $\varphi(u)=-u+\gamma_1u^2+\gamma_2u^3$ in $\mathbb{R}^n(n\leq 3)$, they obtained the global existence and blow-up of classical solutions, and pointed out that the sign of γ_2 is crucial to the global existence of solutions. Grinfeld and Novick-Cohen [5] also studied the viscous Cahn–Hilliard equation. In their paper, a Morse decomposition of the stationary solutions of the 1D viscous Cahn–Hilliard equation was established by explicit energy calculations, and the global attractor for the viscous Cahn–Hilliard equation was also considered. Carvalho and Dlotko [6] studied the following generalized viscous Cahn–Hilliard equation with the first value boundary condition in $W_0^{1,p}(\Omega)$,

$$(1-\nu)u_t=-\Delta(\Delta u+f(u)-\nu u_t). \tag{4}$$

* Corresponding author. Tel.: +86 43185166425; fax: +86 43185188182.
E-mail address: zxp032@gmail.com (X. Zhao).

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In the critical growth case, they proved that the above problem is locally well posed and obtained a bootstrapping procedure showing that the solutions are classical. For $p = 2$ and almost critical dissipative nonlinearities they proved global well posedness, existence of global attractors in $H_0^1(\Omega)$. Finally, they obtained a result on continuity of regular attractors which shows that, if $n = 3, 4$, the attractor of the Cahn–Hilliard equation coincides (in a sense to be specified) with the attractor for the corresponding semilinear heat equation. We also noticed that some investigations of the viscous Cahn–Hilliard equation were studied, such as in [7–11].

The optimal control plays an important role in modern control theories, and has a wider application in modern engineering. Two methods are used for studying control problems in PDE: one is using a low model method, and then changing to an ODE model [12]; the other is using a quasi-optimal control method [13]. No matter which one is chosen, it is necessary to prove the existence of optimal solution according to the basic theory [14].

Many papers have already been published to study the control problems of nonlinear parabolic equations. In 1991, Yong and Zheng [15] considered the feedback stabilization and optimal control of the Cahn–Hilliard equation in a bounded domain with smooth boundary. In the papers wrote by Sang-Uk Ryu and Atsushi Yagi [16,17], the optimal control problems of Keller–Segel equations and adsorbate-induced phase transition model were considered. Their techniques are based on the energy estimates and the compact method. They established various a priori estimates for the solutions of equations to show that the classical compact method described systematically by Lions (see [18]) is available. Tian et al. [19–22] also studied the optimal control problems for parabolic equations, such as viscous Camassa–Holm equation, viscous Degasperis–Procesi equation, viscous Dullin–Gottwald–Holm equation and so on. There is a great deal of literature concerned with the optimal control problem for parabolic equations, for more recent result, we refer the reader to [23–25] and the references therein.

In this research, we are concerned with the distributed optimal control problem

$$\min J(y, \bar{\omega}) = \frac{1}{2}\|Cy - z\|_S^2 + \frac{\delta}{2}\|\bar{\omega}\|_{L^2(Q_0)}^2, \tag{5}$$

subject to

$$\begin{cases} y_t - \dfrac{\gamma}{k} y_{xx} + \dfrac{\gamma}{k} u_{xx} - \varphi(u)_{xx} = B^*\bar{\omega}, & (x, t) \in \Omega \times (0, T), \\ u(x, t) = u_{xx}(x, t) = 0, & x \in \partial\Omega, \\ y(0) = y_0, & y_0 \in H, \end{cases} \tag{6}$$

where $\gamma > 1 + \frac{2}{\pi^4}$, $y = u - ku_{xx}$. Clearly, the control target is to match the given desired state z by adjusting the body force $\bar{\omega}$ in a control volume $Q_0 \subseteq Q = (0, T) \times \Omega$ in the L^2-sense.

Now, we introduce some notations that will be used throughout the paper.

For fixed $T > 0$, we set $\Omega = (0, 1)$ and $Q = \Omega \times (0, T)$, we also let $Q_0 \subseteq Q$ be an open set with positive measure, $V = H_0^1(0, 1)$, $H = L^2(0, 1)$, $V^* = H^{-1}(0, 1)$ and $H^* = H$. Then

$$V \hookrightarrow H = H^* \hookrightarrow V^*,$$

each embedding being dense. The extension operator $B^* \in \mathcal{L}(L^2(Q_0), L^2(V^*))$ is introduced as

$$B^* q = \begin{cases} q, & q \in Q_0, \\ 0, & q \in Q \setminus Q_0. \end{cases}$$

Throughout this paper, we supply H with the inner product $(\cdot, \cdot)$ and the norm $\|\cdot\|$, we also defined a new space $W(0, T; V)$ as

$$W(0, T; V) = \{y : y \in L^2(0, T; V),\ y_t \in L^2(0, T; V^*)\},$$

which is a Hilbert space endowed with common inner product.

This paper is organized as follows. In the next section, we prove the existence of weak solution to the viscous Cahn–Hilliard equation in a special space. We also discuss the relation among the norms of weak solution, initial value and control item. In Section 3, we consider the optimal control problem of the viscous Cahn–Hilliard equation and prove the existence of optimal solution.

In the following, the letters $c, c_i,\ c_i',\ (i = 1, 2, \ldots)$ will always denote positive constants different in various occurrences.

2. Existence of weak solution

In this section, we shall prove the existence of a weak solution for the following equation

$$y_t - \frac{\gamma}{k} y_{xx} + \frac{\gamma}{k} u_{xx} - \varphi(u)_{xx} = B^*\bar{\omega}, \tag{7}$$

with the boundary conditions

$$u(0, t) = u(1, t) = u_{xx}(0, t) = u_{xx}(1, t) = 0, \tag{8}$$