Nonlinear Analysis 74 (2011) 6696–6711

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Blow up oscillating solutions to some nonlinear fourth order differential equations

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ARTICLE INFO

Article history:
Received 8 June 2011
Accepted 28 June 2011
Communicated by Enzo Mitidieri

MSC:
34C10
34A12
65L05

Keywords:
Finite time blow up
Oscillations

ABSTRACT

We give strong theoretical and numerical evidence that solutions to some nonlinear fourth order ordinary differential equations blow up in finite time with infinitely many wild oscillations. We exhibit an explicit example where this phenomenon occurs. We discuss possible applications to biharmonic partial differential equations and to the suspension bridges model. In particular, we give a possible new explanation of the collapse of bridges.

1. Introduction

In this paper, we study the differential equation

$$w''''(s) + kw''(s) + f(w(s)) = 0 \quad (s \in \mathbb{R}) \tag{1}$$

where $k \in \mathbb{R}$ and f is a locally Lipschitz function.

This equation arises in several contexts. With no hope of being exhaustive, let us mention some models which lead to (1). When k is negative (1) is known as the extended Fisher–Kolmogorov equation, whereas when k is positive it is referred to as the Swift–Hohenberg equation; see [1]. Eq. (1) arises in the dynamic phase-space analogy of a nonlinearly supported elastic strut [2] and serves as a model of pattern formation in many physical, chemical or biological systems; see [3,4] and references therein. It may also be used to investigate localization and spreading of deformation of a strut confined by an elastic foundation [5]. A particularly interesting model concerns traveling waves in a suspension bridge; see [6–8] and Section 3.1. Eq. (1) also arises from a suitable transformation of some biharmonic pde's; see [9,10] and Section 3.2. Last but not least, we mention the important book by Peletier–Troy [1] where one can find many other physical models, a survey of existing results, and further references.

The purpose of the present paper is to contribute to a better understanding of possible finite time blow up for solutions to (1) when the nonlinearity f satisfies

$$f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}), \quad f(t)\,t > 0 \quad \text{for every } t \in \mathbb{R} \setminus \{0\}. \tag{2}$$

Further assumptions on f will be needed in the following. We are interested in necessary and/or sufficient conditions for local solutions to be global.

The starting point of our study are the following results proved in [10].

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0362-546X/$ – see front matter
doi:10.1016/j.na.2011.06.049

Proposition 1. *Let $k \in \mathbb{R}$ and assume that f satisfies (2).*

(i) *If a local solution w to (1) blows up at some finite $R \in \mathbb{R}$, then*

$$\liminf_{s\to R} w(s) = -\infty \quad \text{and} \quad \limsup_{s\to R} w(s) = +\infty. \tag{3}$$

(ii) *If f also satisfies*

$$\limsup_{t\to+\infty} \frac{f(t)}{t} < +\infty \quad \text{or} \quad \limsup_{t\to-\infty} \frac{f(t)}{t} < +\infty, \tag{4}$$

then any local solution to (1) exists for all $s \in \mathbb{R}$.

Proposition 1 (i) states that, under the sole assumption (2), the only way that the finite time blow up can occur is with wide oscillations of the solution. Also note that if *both* the conditions in (4) are satisfied then the global existence follows from classical theory of ode's. But (4) merely requires that f is "one-sided at most linear". Most of the nonlinearities f in the models mentioned before satisfy (4) which, in particular, includes the cases where f is either concave or convex. Several examples exhibited in [10] show that if (2) is violated, then local solutions to (1) may blow up in finite time. Motivated by Berchio et al. [10, Problem 3] we are here interested in studying the case where (2) is satisfied but (4) fails.

In next section, we bring strong evidence that if (4) fails then the solution to (1) blows up in finite time. Our first theoretical result (see Theorem 2) states that, under suitable assumptions on f and k, the solution to (1) exhibits "wide and thinning oscillations". By this we mean that the altitude of the oscillation increases and tends to infinity whereas its cycle (the distance between two consecutive zeros of the solution) decreases and tends to zero. Clearly, this *does not* prove that blow up occurs in finite time but, at least, it gives a strong hint in this direction. Our second theoretical result (see Theorem 3) gives an explicit example where the finite time blow up with wide and thinning oscillations indeed occurs. As far as we are aware, this is the first example which exhibits this phenomenon. As we shall see in Section 3.2, this statement has deep applications in some semilinear biharmonic problems. Moreover, in Section 3.3 we show that this kind of blow up phenomenon is typical of (at least) fourth order problems such as (1) since it does not occur in related lower order equations with f satisfying (4). Finite time blow up in more general situations is supported by our numerical results. In Section 4, we describe the numerical procedure used to obtain the plots and tables displayed in the next section. We show there that the solution to (1) seems to blow up in finite time, both by plotting its wide oscillations and by approximating the sequence of its zeros.

We believe that this kind of finite time blow up can give an explanation to the celebrated collapse of the Tacoma Narrows Bridge [11]. We discuss in detail our point of view in Section 3.1.

This paper is organized as follows. In Section 2, we state our main theoretical results and we describe our main numerical results. In Sections 3.1 and 3.2, we discuss the above-mentioned suspension bridges model and some possible applications of (1) to biharmonic coercive elliptic partial differential equations; in particular, the latter enables us to prove Theorem 3. In Section 3.3, we show that the phenomena we find for the fourth order Eq. (1) do not hold for lower order equations. In Section 4, we explain how we obtained the numerical results. Sections 5 and 6 are devoted to the proof of Theorem 2. Finally, in Section 7, we give further numerical results which enable us to comment the assumptions on f and k.

2. Main results

Assume that f satisfies

$$f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}), \quad \text{there exists } \lambda, \delta, \gamma > 0 \text{ such that } f(t)t \geq \delta t^2 + \lambda |t|^{2+\gamma} \quad \text{for every } t \in \mathbb{R} \tag{5}$$

and note that (5) strengthens (2). Consider (1) with $k = 0$:

$$w''''(s) + f(w(s)) = 0 \quad (s \in \mathbb{R}). \tag{6}$$

We prove that if f is superlinear then any local solution w to (6) satisfying suitable initial conditions at $s = 0$ has infinitely many oscillations which tend to enlarge and concentrate on small intervals.

Theorem 2. *Assume that f satisfies (5). Let w be a local solution to (6) in a neighborhood of $s = 0$ such that*

$$w'(0)w''(0) - w(0)w'''(0) > 0. \tag{7}$$

Let $R \in (0, +\infty]$ denote the supremum of the maximal interval of continuation of w. Then there exists an increasing sequence $\{s_j\}_{j\in\mathbb{N}}$ such that:

(i) *$s_j \nearrow R$ as $j \to \infty$;*
(ii) *$w(s_j) = 0$ and w has constant sign in (s_j, s_{j+1}) for all $j \in \mathbb{N}$;*
(iii) *$\lim_{j\to\infty}(s_{j+1} - s_j) = 0$;*
(iv) *$\max_{s\in[s_j,s_{j+1}]} |w(s)| \to +\infty$ as $j \to \infty$.*