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# Multiple solutions of Sturm–Liouville boundary value problem via lower and upper solutions and variational methods☆

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ABSTRACT

In this paper, we prove the existence of multiple solutions for second order Sturm–Liouville boundary value problem

$$\begin{cases} -Lu = f(x, u), & x \in [0, 1] \\ R_1(u) = 0, & R_2(u) = 0, \end{cases}$$

where $Lu = (p(x)u')' - q(x)u$ is a Sturm–Liouville operator, $R_1(u) = \alpha u'(0) - \beta u(0)$, $R_2(u) = \gamma u'(1) + \sigma u(1)$. The technical approach is fully based on lower and upper solutions and variational methods. The interesting point is that the existence of four solutions and seven solutions is given.



## 1. Introduction

In recent years, there have been many papers studying the existence of solutions for boundary value problems, please refer to [1–8]. Agarwal et al. [9], Anuradha et al. [10], Erbe and Wong [11], Ge and Ren [12], Sun and Zhang [13], Zhang and Sun [14] have studied positive solutions of Sturm–Liouville boundary value problem by using fixed point theorem. Mao and Zhang [15] studied the existence of solutions for Kirchhoff type problems by using minimax methods and invariant sets of descending flow. Zhang and Perera [16] obtained the existence of positive, negative and sign-changing solutions of a class of nonlocal quasilinear elliptic boundary value problems using variational methods and invariant sets of descending flow. In papers [17,18], the existence of positive, negative and sign-changing solutions for asymptotically linear three-point boundary value problems was studied by using the topological degree theory and the fixed point index theory when the nonlinear term $f$ is continuous and strictly increasing. In paper [39], Han and Li studied the existence of solutions for fourth order boundary value problem by using the critical point theory and the supersolution and subsolution method. Bonanno and Riccobono [4], Bonanno and Molica Bisci [19], Ricceri [20,21], Averna and Bonanno [22], Tian and Ge [23–26] studied positive solutions and multiple solutions for boundary value problems by using variational methods.

In papers [13,27], Sun and Zhang studied Sturm–Liouville boundary value problem

$$\begin{cases} -(L\varphi)(x) = h(x)f(\varphi(x)), & 0 < x < 1, \\ R_1(\varphi) = \alpha_1\varphi(0) + \beta_1\varphi'(0) = 0, \\ R_2(\varphi) = \alpha_2\varphi(1) + \beta_2\varphi'(1) = 0, \end{cases}$$

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where $(L\varphi)(x) = (p(x)\varphi'(x))' + q(x)\varphi(x)$, $p > 0$, $q \le 0$, $h$ is allowed to be singular at both $x = 0$, $x = 1$, $f$ is not necessary to be nonnegative. By using the topological degree theory, the existence results of nontrivial solutions and positive solutions are given.

In paper [4], Bonanno and Riccobono have studied Sturm–Liouville boundary value problem

$$\begin{cases} -(\rho\Phi_p(x'))' + s\Phi_p(x) = \lambda f(t, x), & t \in [a, b], \\ \alpha x'(a) - \beta x(a) = A, & \gamma x'(b) + \sigma x(b) = B, \end{cases} \tag{1.1}$$

where $p > 1$, $\Phi_p(x) = |x|^{p-2}x$, $\rho, s \in L^\infty([a, b])$, with $\operatorname{ess\,inf}_{[a,b]} \rho > 0$ and $\operatorname{ess\,inf}_{[a,b]} s > 0$, $A, B \in R$, $\alpha, \beta, \sigma, \gamma > 0$, $f : [a, b] \times R \to R$ an $L^1$-Carathédory function and $\lambda$ a positive real parameter. By using three-critical-points theorem [38], problem (1.1) has at least three weak solutions when $f$ satisfies sublinear condition $\int_0^\xi f(t, s)ds \le \mu(t)(1 + |\xi|^l)$, $l < p$ and a local restriction condition, $\lambda$ is in some suitable interval.

In paper [19], Bonanno and Molica Bisci proved a theorem on the existence of an unbounded (in $W_0^{1,2}[0, 1]$) sequence of weak solutions for the following Sturm–Liouville boundary value problem

$$(p(t)u')' + q(t)u = f(u), \quad u(0) = u(1) = 0,$$

where $p, q \in L^\infty[0; 1]$, $\lambda > 0$ is a parameter and $f : R \to R$ is almost everywhere continuous (i.e., the Lebesgue measure of its set of discontinuities is zero) and locally essentially bounded. To this end, they give a new proof for a properly modified version of the result by Marano and Motreanu [J. Differential Equations 182 (1) (2002) 108–120; MR1912071 (2004b:49015)] on infinitely many critical points of non-differentiable functions.

Our aim of this paper is to apply lower and upper solutions and variational methods into Sturm–Liouville boundary value problem and obtain further the existence of multiple solutions. We, in this paper, study the existence of multiple solutions for second order Sturm–Liouville boundary value problem

$$\begin{cases} -Lu = f(x, u), & x \in [0, 1] \\ R_1(u) = 0, & R_2(u) = 0, \end{cases} \tag{1.2}$$

where $Lu = (p(x)u')' - q(x)u$ is a Sturm–Liouville operator, $R_1(u) = \alpha u'(0) - \beta u(0)$, $R_2(u) = \gamma u'(1) + \sigma u(1)$, $\alpha, \beta, \sigma, \gamma \ge 0$, $\alpha^2 + \beta^2 > 0$, $\gamma^2 + \sigma^2 > 0$, $p, q \in C^1[0, 1]$, $p, q > 0$, $(x, u) \to f(x, u)$ is continuous in $[0, 1] \times R$, and Lipschitz continuous for $u$ uniformly in $x \in [0, 1]$. We obtain the existence of four solutions and seven solutions for problem (1.2). Besides, we give more accurate characterization of the specific scope of solutions. The results are different from those in the literature. Our main idea comes from [28–31].

Contrast with the results in paper [26], the existence of three solutions are obtained in [26], where one solution is sign-changing. In this paper, at least four solutions and seven solutions are obtained. When the upper solution is positive, and lower solution is nonpositive, the sign-changing solution is obtained.

The paper is organized as follows: In Sections 2 and 3 we first discuss the case $\alpha, \gamma > 0$ in Sturm–Liouville boundary condition $R_1(u) = 0$, $R_2(u) = 0$ for the convenience of expressing the functional $\Phi(u)$. In Section 2, we give some basic knowledge and related lemmas. In Section 3, we prove the existence of four solutions by applying descending flow and lower and upper solutions method. In Section 4, we prove the existence of seven solutions by applying Hofer's variational methods and lower and upper solutions method. In Section 5, we discuss the case $\alpha, \gamma \ge 0$ in Sturm–Liouville boundary condition $R_1(u) = 0$, $R_2(u) = 0$ and give examples to illustrate the main results.

## 2. Basic knowledge with $\alpha, \gamma \neq 0$

Let the space $H = H^1[0, 1]$ be the Sobolev space. We define new inner product of $H$ as follows

$$(u, v) = \int_0^1 p(x)u'(x)v'(x) + q(x)u(x)v(x)dx + \frac{\sigma p(1)}{\gamma}u(1)v(1) + \frac{\beta p(0)}{\alpha}u(0)v(0).$$

The inner product induces the norm

$$\|u\|_H = \left(\int_0^1 p(x)|u'(x)|^2 + q(x)|u(x)|^2dx + \frac{\sigma p(1)}{\gamma}|u(1)|^2 + \frac{\beta p(0)}{\alpha}|u(0)|^2\right)^{\frac{1}{2}},$$

which is equivalent to the usual one. The norm $\|u\|_H$ is a part of the functional $\Phi$ ((2.1)), which makes it convenient to estimate $\Phi$.

We define the space $X = C^1[0, 1]$ with the norm $\|u\|_X = \max\{\max_{x\in[0,1]} |u(x)|, \max_{x\in[0,1]} |u'(x)|\}$. Clearly $X$ is densely imbedded in $H$.

Define the functional $\Phi : H \to R$ by

$$\Phi(u) = \frac{1}{2}\int_0^1 p(x)|u'(x)|^2 + q(x)|u(x)|^2dx + \frac{\sigma p(1)}{2\gamma}|u(1)|^2 + \frac{\beta p(0)}{2\alpha}|u(0)|^2 - \int_0^1 F(x, u(x))dx, \tag{2.1}$$

where $F(x, u) = \int_0^u f(x, s)ds$.