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Coincidence and common fixed point results in partially ordered cone metric spaces and applications to integral equations

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ABSTRACT

In this paper, coincidence and common fixed point results are established in a partially ordered cone metric space. An application of our results obtained to prove the existence of a common solution to integral equations is presented.

1. Introduction

Fixed point theory has fascinated many researchers since 1922 with the celebrated Banach's fixed point theorem. There exists a vast literature on the topic and this is a very active field of research at present. A self-map T on a metric space X is said to have a fixed point $x \in X$ if $Tx = x$. Theorems concerning the existence and properties of fixed points are known as fixed point theorems. Such theorems are very important tools for proving the existence and eventually the uniqueness of the solutions to various mathematical models (integral and partial differential equations, variational inequalities, ...).

Huang and Zhang [1] reintroduced the concept of a cone metric space, where every pair of elements is assigned to an element of a Banach space equipped with a cone which induces a natural partial order. In the same work, they investigated the convergence in cone metric spaces, introduced the notion of their completeness, and proved some fixed point theorems for mappings satisfying different contractive conditions on these spaces. After that, fixed point results for cone metric spaces were studied by many other authors. Refs. [2–16] are some works in this line of research.

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The weak contraction principle was first given by Alber et al. for Hilbert spaces [17] and subsequently extended to metric spaces by Rhoades [18]. After that, fixed point problems involving weak contractions and mappings satisfying weak contraction type inequalities were considered in several works like [19–21,6,22]. In particular, in cone metric spaces the weak contraction principle was extended by Choudhury and Metiya in [21,6].

Recently, there have been so many exciting developments in the field of existence of fixed point in partially ordered sets. The first result in this direction was given by Ran and Reurings [23] where they extended the Banach contraction principle in partially ordered sets with some applications to matrix equations. The obtained result by Ran and Reurings was further extended and refined in [12]. For more details on fixed point theory in partially ordered sets, we refer the reader to [24–29,12,30,31] and the references cited therein.

In this paper we establish some coincidence and common fixed point results for three self-mappings on a partially ordered cone metric space satisfying a weak contractive condition of Choudhury–Metiya type. Presented theorems extend very recent results obtained by Choudhury and Metiya in [6]. We use one of our obtained results to prove an existence theorem of a common solution of integral equations.

2. Preliminaries

Let E be a real Banach space with respect to a given norm $\|\cdot\|_E$ and 0_E is the zero vector of E.

Definition 2.1. A non-empty subset P of E is called a cone if the following conditions hold:

(i) P is closed and $P \neq \{0_E\}$;
(ii) $a, b \in \mathbb{R},\ a, b \geq 0,\ x, y \in P \Longrightarrow ax + by \in P$;
(iii) $x \in P,\ -x \in P \Longrightarrow x = 0_E$.

Given a cone $P \subset E$, a partial ordering $\leq_E$ with respect to P is naturally defined by $x \leq_E y$ if and only if $y - x \in P$, for $x, y \in E$. We shall write $x <_E y$ to indicate that $x \leq_E y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int}P$, where int$P$ denotes the interior of P.

The cone P is said to be normal if there exists a real number $K > 0$ such that for all $x, y \in E$,

$$0_E \leq_E x \leq_E y \Longrightarrow \|x\|_E \leq K\|y\|_E.$$

The least positive number K satisfying the above statement is called the normal constant of P.

The cone P is called regular if every increasing sequence which is bounded from above is convergent, that is, if $\{x_n\}$ is a sequence such that

$$x_1 \leq_E x_2 \leq_E \cdots \leq_E x_n \leq_E \cdots \leq_E y$$

for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\|_E \to 0$ as $n \to +\infty$. Equivalently, the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone.

In the following we always suppose that E is a real Banach space with cone P with int$P \neq \emptyset$ and $\leq_E$ is the partial ordering in E with respect to P.

Definition 2.2 (*[6]*)**.** Let $\psi : P \to P$ be a given function.

(i) We say that ψ is strongly monotone increasing if for $x, y \in P$, we have

$$x \leq_E y \Longleftrightarrow \psi(x) \leq_E \psi(y).$$

(ii) ψ is said to be continuous at $x_0 \in P$ if for any sequence $\{x_n\}$ in P, we have

$$\|x_n - x_0\|_E \to 0 \Longrightarrow \|\psi(x_n) - \psi(x_0)\|_E \to 0.$$

Definition 2.3 (*[1]*)**.** Let X be a non-empty set and $d : X \times X \to P$ satisfies

(i) $d(x, y) = 0_E$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \leq_E d(x, z) + d(z, y)$ for all $x, y, z \in E$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 2.4 (*[1]*)**.** Let (X, d) be a cone metric space, $\{x_n\}$ is a sequence in X and $x \in X$.

(i) If for every $c \in E$ with $0_E \ll c$, there is $N \in N$ such that $d(x_n, x) \ll c$ for all $n \geq N$, then $\{x_n\}$ is said to be convergent to x. This limit is denoted by $\lim_{n\to+\infty} x_n = x$ or $x_n \to x$ as $n \to +\infty$.
(ii) If for every $c \in E$ with $0_E \ll c$, there is $N \in N$ such that $d(x_n, x_m) \ll c$ for all $n, m > N$, then $\{x_n\}$ is called a Cauchy sequence in X.
(iii) If every Cauchy sequence in X is convergent in X, then (X, d) is called a complete cone metric space.