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The existence and exponential stability of semilinear functional differential equations with random impulses under non-uniqueness

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ABSTRACT

In this paper, the existence and exponential stability of mild solutions of semilinear differential equations with random impulses are studied under non-uniqueness in a real separable Hilbert space. The results are obtained by using the Leray–Schauder alternative fixed point theorem.

1. Introduction

Many evolution processes from fields such as physics, population dynamics, aeronautics, economics, telecommunications and engineering are characterized by the fact that they undergo an abrupt change of state at certain moments of time between intervals of continuous evolution. The durations of these changes are often negligible compared to the total duration of the process, since they act instantaneously in the form of impulses. It is now being recognized that the theory of impulsive differential equations is not only richer than the corresponding theory of differential equations but also represents a more natural framework for mathematical modeling of many real-world phenomena; see [1–5] and the references therein.

The impulses exist at fixed times or at random times, i.e., they are deterministic or random. Many papers have investigated the qualitative properties of fixed-type impulses; see [1–5] and the references therein. There are only a few papers that have studied random-type impulses. Wu and Meng [6] first introduced random impulsive ordinary differential equations and investigated the boundedness of solutions to these models by Liapunov's direct method in [6]. In [7], Wu and Duan discussed the oscillation, stability and boundedness of solutions to the model by comparing the solutions of this system with the corresponding non-impulsive differential system. In [8], Wu et al. discussed the existence and uniqueness in the mean square of solutions to certain random impulsive differential systems employing the Cauchy–Schwarz inequality, Lipschitz condition and techniques in stochastic analysis. In [9], Wu et al. first introduced random impulsive functional differential equations and considered the p-moment stability of solutions to these models using Liapunov's function coupled

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with the Razumikhin technique. Then, Wu et al. [8] considered the almost sure stability of solutions to random impulsive functional differential equations by Liapunov's function coupled with the Razumikhin technique in [10].

In [11], Burton and Zhang studied the existence and asymptotic stability through fixed point theory. Luo [12] studied the exponential stability and almost sure exponential stability in the pth mean of mild solutions of stochastic differential equations by means of the contraction mapping principle. In [12], the system possesses a unique solution by the Lipschitz condition. In this paper, the existence and exponential stability results of random impulsive semilinear differential equations are studied. To the best of our knowledge, this is the first paper which studies the existence and exponential stability of random impulsive differential system by means of fixed point theory. Under our assumption, the system guarantees non-uniqueness; we utilize the technique developed in [13–15,8].

The paper is organized as follows. Some preliminaries are presented in Section 2. In Section 3, we investigate the existence of mild solutions of semilinear differential equations with random impulses by using the Leray–Schauder alternative fixed point theory. An interesting feature of this method is that this yields simultaneously the existence and maximal interval of existence. In Section 4, we study the exponential stability of mild solutions of semilinear differential equations with random impulses through the same fixed point theory, and finally, in Section 5, we construct some applications for our abstract results.

2. Preliminaries

Let X be a real separable Hilbert space and Ω a nonempty set. Assume that τ_k is a random variable defined from Ω to $D_k \overset{def.}{=} (0, d_k)$ for $k = 1, 2, \ldots$, where $0 < d_k < +\infty$. Furthermore, assume that τ_i and τ_j are independent from each other as $i \neq j$ for $i, j = 1, 2, \ldots$. Let $\tau, T \in \Re$ be two constants satisfying $\tau < T$. For the sake of simplicity, we denote

$$\Re^+ = [0, +\infty); \qquad \Re_\tau = [\tau, +\infty).$$

We consider semilinear differential equations with random impulses of the form

$$\begin{cases} x'(t) = Ax(t) + f(t, x_t), & t \neq \xi_k,\ t \geq \tau, \\ x(\xi_k) = b_k(\tau_k)x(\xi_k^-), & k = 1, 2, \ldots, \\ x_{t_0} = \varphi, \end{cases} \tag{2.1}$$

where A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $S(t)$ in X; the functional $f : \Re_\tau \times \hat{C} \to X$, $\hat{C} = \hat{C}([-r, 0], X)$, is the set of piecewise continuous functions mapping $[-r, 0]$ into X with some given $r > 0$; x_t is a function when t is fixed, defined by $x_t(s) = x(t + s)$ for all $s \in [-r, 0]$; $\xi_0 = t_0$ and $\xi_k = \xi_{k-1} + \tau_k$ for $k = 1, 2, \ldots$; here $t_0 \in \Re_\tau$ is an arbitrary given real number. Obviously, $t_0 = \xi_0 < \xi_1 < \xi_2 < \cdots < \lim_{k\to\infty} \xi_k = \infty$; $b_k : D_k \to \Re$ for each $k = 1, 2, \ldots$; $x(\xi_k^-) = \lim_{t\uparrow\xi_k} x(t)$ according to their paths with the norm $\|x\|_t = \sup_{t-r\leq s\leq t} |x(s)|$, for each t satisfying $\tau \leq t \leq T$, $\|\cdot\|$ is any given norm in X; φ is a function defined from $[-r, 0]$ to X.

Denote $\{B_t, t \geq 0\}$ the simple counting process generated by $\{\xi_n\}$, that is, $\{B_t \geq n\} = \{\xi_n \leq t\}$, and denote $\mathcal{F}_t$ the σ-algebra generated by $\{B_t, t \geq 0\}$. Then $(\Omega, P, \{\mathcal{F}_t\})$ is a probability space. Let $L_2 = L_2(\Omega, \mathcal{F}_t, X)$ denote the Hilbert space of all $\mathcal{F}_t$-measurable square integrable random variables with values in X.

Let Γ denote the Banach space $\Gamma\big([t_0 - r, T], L_2\big)$, the family of all $\mathcal{F}_t$-measurable, $\hat{C}$-valued random variables ψ with the norm

$$\|\psi\|_\Gamma = \left(\sup_{t_0\leq t\leq T} E\|\psi\|_t^2 \right)^{1/2}.$$

Let $L_2^0(\Omega, \Gamma)$ denote the family of all $\mathcal{F}_0$-measurable, Γ-valued random variable φ.

Definition 2.1. A semigroup $\{S(t); t \geq t_0\}$ is said to be exponentially stable if there are positive constants $M \geq 1$ and $\gamma > 0$ such that $\|S(t)\| \leq M e^{-\gamma(t-t_0)}$ for all $t \geq t_0$, where $\|\cdot\|$ denotes the operator norm in $\mathcal{L}(X)$ (the Banach algebra of bounded linear operators from X into X). A semigroup $\{S(t),\ t \geq t_0\}$ is said to be uniformly bounded if $\|S(t)\| \leq M$ for all $t \geq t_0$, where $M \geq 1$ is some constant. If $M = 1$, then the semigroup is said to be a contraction semigroup.

Definition 2.2. A map $f : [\tau, T] \times \hat{C} \to X$ is said to be L^2-Carathéodory, if

(i) $t \to f(t, u)$ is measurable for each $u \in \hat{C}$;
(ii) $u \to f(t, u)$ is continuous for almost all $t \in [\tau, T]$;
(iii) for each positive integer $m > 0$, there exists $\alpha_m \in L^1([\tau, T], \Re^+)$ such that

$$\sup_{\|u\|_\Gamma\leq m} \|f(t, u)\|^2 \leq \alpha_m(t), \quad \text{for } t \in [\tau, T],\ \text{a.e.}$$

Definition 2.3. For a given $T \in (t_0, +\infty)$, a stochastic process $\{x(t) \in \Gamma,\ t_0 - r \leq t \leq T\}$ is called a mild solution to Eq. (2.1) in $(\Omega, P, \{\mathcal{F}_t\})$, if

(i) $x(t) \in X$ is $\mathcal{F}_t$-adapted for $t \geq t_0$;