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Lipschitzian stability of parametric variational inequalities over generalized polyhedra in Banach spaces

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ABSTRACT

This paper concerns the study of solution maps to parameterized variational inequalities over generalized polyhedra in reflexive Banach spaces. It has been recognized that generalized polyhedral sets are significantly different from the usual convex polyhedra in infinite dimensions and play an important role in various applications to optimization, particularly to generalized linear programming. Our main goal is to fully characterize robust Lipschitzian stability of the aforementioned solution maps entirely via their initial data. This is done on the basis of the coderivative criterion in variational analysis via efficient calculations of the coderivative and related objects for the systems under consideration. The case of generalized polyhedra is essentially more involved in comparison with usual convex polyhedral sets and requires developing elaborated techniques and new proofs of variational analysis.

1. Introduction

Parametric variational inequalities are among the most important objects in optimization theory and variational analysis; see, e.g., books [1–6] and the references therein. A breakthrough in their study and applications goes back to the seminal work by Robinson [7,8] who treated them as parametric "generalized equations"

$$0 \in f(p,x) + N(x;\Theta) \quad \text{for all } x \in \Theta, \tag{1.1}$$

where $x \in X$ is the decision variable and $p \in Z$ is the parameter taking values in the corresponding Banach spaces. The "base" mapping $f: Z \times X \to X^*$ in (1.1) takes values in the dual space X^* while the set-valued "field" part $N: X \rightrightarrows X^*$ is the normal cone mapping to a convex set $\Theta \subset X$. By the classical definition of the normal cone in convex analysis with $N(x;\Omega) := \emptyset$ if $x \notin \Theta$, the generalized equation form (1.1) is equivalent to the standard form of variational inequalities: for each $p \in Z$ find $x \in \Theta$ such that

$$\langle f(p,x), x - u\rangle \le 0 \quad \text{whenever } u \in \Theta.$$

It has been well recognized that the generalized equation formalism (1.1) is a convenient model to describe parametric complementarity problems, moving sets of optimal solutions to various optimization and equilibrium problems, KKT systems, and the like; see the references mentioned above and the bibliographies therein.

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Consider the *solution map* $S: Z \rightrightarrows X$ to the parametric variational inequality/generalized equation (1.1) defined by

$$S(p) := \{x \in X | 0 \in f(p,x) + N(x;\Theta)\}. \tag{1.2}$$

The dependence of (1.2) on the parameter variable $p \in Z$ is one of the major issues from the viewpoints of sensitivity and stability analysis of the variational systems under consideration and their applications to parametric and hierarchical optimization, mathematical programs with equilibrium constraints, etc. *Robust Lipschitzian behavior* (i.e., stable with respect to perturbations of the initial data) of the solution map (1.2) and its *quantitative* characteristics are among the most important goals to achieve.

Advanced variational analysis and generalized differentiation offer verifiable pointwise characterizations of such behavior around reference points by computing the exact Lipschitzian moduli via the so-called *coderivatives* of general set-valued mappings; see [3,4,6] and Section 2 for more details. However, implementations of these criteria and their realizations in terms of the initial data of variational systems of type (1.2) are definitely not an easy job in both finite and infinite dimensions, where the latter case creates additional serious complications due to the lack of compactness.

A remarkable class of convex sets is described by *convex polyhedra*

$$\Theta := \{x \in X | \langle x_i^*, x\rangle \le c_i \text{ for } i = 1, \ldots, m\}, \tag{1.3}$$

where $x_i^* \in X^*$ are fixed elements. Significant progress in the study and applications of Lipschitzian stability for parametric variational inequalities (1.1) over polyhedral convex sets (1.3) has been achieved on the basis of coderivative characterizations mainly in finite dimensions [9–13] and quite recently in reflexive Banach spaces [14,15].

The major attention of this paper is paid to robust Lipschitzian stability of parametric variational inequalities over the so-called *generalized polyhedral sets* defined by

$$\Theta := \{x \in X | Ax = b \text{ and } \langle x_i^*, x\rangle \le c_i, \text{ for } i = 1, \ldots, m\} \tag{1.4}$$

and formed by fixed elements $x_i^* \in X^*$, $b \in Y$, $c_i \in \mathbb{R}$ and a linear bounded operator $A: X \to Y$ from X to another Banach space Y.

In contrast to the case of finite-dimensional spaces, the generalized polyhedra (1.4) do not reduce to the usual ones (1.3) in infinite dimensions. The "generalized polyhedral" terminology has been coined in [1], where systems (1.4) were largely investigated from the viewpoint of applications to the generalized linear programs

$$\text{minimize } \langle a, x\rangle \text{ subject to } Ax = b \quad \text{and} \quad \langle x_i^*, x\rangle \le c_i \text{ for } i = 1, \ldots, m \tag{1.5}$$

as well as to problems of concave minimization under generalized polyhedral constraints (1.4). We refer the reader to [16] for the study of (generalized) linear programs in infinite dimensions and their applications to problems in approximation theory, mass-transfer, optimal control, dynamic network, and semi-infinite and infinite programming. Book [17] is particularly devoted to linear semi-infinite programming, while the recent papers [18,19] concern robust stability issues and optimality conditions for semi-infinite and infinite programs with linear inequality constraints.

Among important classes of infinite-dimensional problems that can be written in the general polyhedral form (1.5) but not with merely polyhedral constraints (1.3) we mention discrete-time Markov decision processes with discounted cost, deterministic continuous-time control problems and those of singular stochastic control problems related to Mather's variational principle, etc.; see, e.g., [20–24] for more details and references.

The main goal of this paper is to obtain complete characterizations of the *Lipschitz-like property* (known also as the Aubin property) of solution maps (1.2) to parametric variational inequalities (1.1) over generalized polyhedral sets Θ from (1.4) entirely in terms of the initial data f, A, b, x_i^*, and c_i. The Lipschitz-like/Aubin property has been well recognized in nonlinear analysis as the most natural extension of the classical local Lipschitz continuity to the case of set-valued mappings, with a localization around the reference point of the graph. This property usually accumulates the amount of robust stability needed for the analysis of constraint and variational systems; see [3,4,6] and the references therein.

Similarly to [14] the approach of this paper is based on implementing the coderivative characterizations [3] of the Lipschitz-like property for general set-valued mappings between infinite-dimensional spaces to the case of the solution map (1.2) generated by Θ from (1.4) instead of that from (1.3) as in [14]. It occurs however that the case of generalized polyhedra is significantly more involved and requires essential elaborations, which are done below. Furthermore, some of the results obtained in this paper are new even in the case of usual convex polyhedra in finite and infinite dimensions.

The rest of the paper is organized as follows. Section 2 collects preliminaries from variational analysis and generalized differentiations widely used in what follows.

Sections 3 and 4 are devoted to technical issues of generalized differentiation of undoubted independent interest, which is crucial for employing the coderivative characterizations of robust stability. Namely, in Section 3 we compute the so-called precoderivative of the normal cone mapping $N(\cdot;\Theta)$ over the generalized polyhedron Θ from (1.4) in reflexive Banach spaces. This serves as a building block for computing the coderivative of the mapping $N(\cdot;\Theta)$ by a limiting procedure.

Section 5 contains the main results of the paper on a complete characterization of the Lipschitz-like property of the solution map (1.2) to the underlying variational inequality (1.1) over the generalized polyhedron (1.4) in reflexive Banach spaces. We not only derive necessary and sufficient conditions for this property but also compute the exact bound of Lipschitzian moduli, which provides the most important qualitative characteristics of robust stability entirely in terms of the initial data of the variational system (1.2), (1.4) under consideration.