Nonlinear Analysis 74 (2011) 501–515

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

A Browder degree theory from the Nagumo degree on the Hilbert space of elliptic super-regularization

Athanassios G. Kartsatos *, David Kerr

Department of Mathematics, University of South Florida, Tampa, FL 33620-5700, USA

ARTICLE INFO

Article history:
Received 24 November 2009
Accepted 4 September 2010

MSC:
primary 47H14
secondary 47H05
47H11

Keywords:
Browder degree theory
Nagumo degree theory
Skrypnik degree theory
Maximal monotone operator
Bounded demicontinuous operator of type (S_+)
Compact operator
Elliptic super-regularization

ABSTRACT

Let X be a real reflexive separable locally uniformly convex Banach space with locally uniformly convex dual space X^*. Let $Q : H \to X$ be a linear compact injection, according to Browder and Ton, such that $\overline{Q(H)} = X$, where H is a real separable Hilbert space. A degree mapping d on X is constructed from the Nagumo degree d_{NA} on H by

$$d(T + f, G, 0) := \lim_{t\to 0} d_{NA}\left(I + \frac{1}{t}Q^*(T_t + f)Q, Q^{-1}G, 0\right),$$

where $G \subset X$ is open and bounded, T_t is the resolvent $(T^{-1} + tJ^{-1})^{-1}$ of a strongly quasibounded maximal monotone operator $T : X \supset D(T) \to 2^{X^*}$ with $0 \in T(0)$, and $f : \overline{G} \to X^*$ is demicontinuous, bounded and of type (S_+). A "range of sums" result is also given, using the Skrypnik degree theory, in order to further exhibit the methodology of "elliptic super-regularization".

1. Introduction — preliminaries

Unless otherwise stated, the symbol X stands for a real reflexive locally uniformly convex Banach space with locally uniformly convex dual space X^*. The symbols $\|\cdot\|_X$ and $\|\cdot\|_{X^*}$ stand for the norms of X and X^*, respectively and $J : X \to X^*$ is the normalized duality mapping. The symbol H is reserved for a real separable Hilbert space. In what follows, "continuous" means "strongly continuous" and the symbol "$\to$" ("$\rightharpoonup$") means strong (weak) convergence. Also, "demicontinuous" means strong-to-weak continuous. The symbol $\mathcal{R}$ ($\mathcal{R}_+$) stands for the set $(-\infty, \infty)$ ($[0, \infty)$) and the symbols ∂D, $\overset{\circ}{D}$, $\overline{D}$, denote the strong boundary, interior and closure of the set D, respectively. We denote by $B_r(0)$ the open ball of X, or X^*, or H with center at zero and radius $r > 0$.

For an operator $T : X \to 2^{X^*}$ we denote by $D(T)$ the effective domain of T, i.e. $D(T) = \{x \in X : Tx \neq \emptyset\}$. We denote by $G(T)$ the graph of T, i.e. $G(T) = \{(x, y) : x \in D(T),\ y \in Tx\}$. An operator $T : X \supset D(T) \to 2^{X^*}$ is called "monotone" if for every $x, y \in D(T)$ and every $u \in Tx,\ v \in Ty$ we have

$$\langle u - v, x - y\rangle \geq 0.$$

A monotone operator T is "maximal monotone" if $G(T)$ is maximal in $X \times X^*$, when $X \times X^*$ is partially ordered by inclusion. In our setting, a monotone operator T is maximal if and only if $R(T + \lambda J) = X^*$ for all $\lambda \in (0, \infty)$. If T is maximal monotone, then the operator $T_t \equiv (T^{-1} + tJ^{-1})^{-1} : X \to X^*$ is bounded, demicontinuous, maximal monotone and such that $T_t x \rightharpoonup T^{\{0\}}x$

* Corresponding author. Tel.: +1 813 974 2643; fax: +1 813 974 2700.
E-mail addresses: hermes@usf.edu, hermes@math.usf.edu (A.G. Kartsatos), kerr@eckerd.edu (D. Kerr).

0362-546X/$ – see front matter
doi:10.1016/j.na.2010.09.005

as $t \to 0^+$ for every $x \in D(T)$, where $T^{\{0\}}x$ denotes the element $y^* \in Tx$ of minimum norm, i.e. $\|T^{\{0\}}x\| = \inf\{\|y^*\| : y^* \in Tx\}$. In our setting, this infimum is always attained and $D(T^{\{0\}}) = D(T)$. Also, $T_t x \in TJ_t x$, where $J_t \equiv I - tJ^{-1}T_t : X \to X$ and satisfies $\lim_{t\to 0} J_t x = x$ for all $x \in \overline{\mathrm{co}\, D(T)}$, where co A denotes the convex hull of the set A. In addition, $x \in D(T)$ and $t_0 > 0$ imply $\lim_{t\to t_0} T_t x = T_{t_0} x$. The operators T_t, J_t were introduced by Brézis, Crandall and Pazy in [1]. For their basic properties, we refer the reader to [1] as well as Pascali and Sburlan [2, pp. 128–130]. In our setting, the duality mapping J is single valued, injective, surjective and bicontinuous.

Unless otherwise stated, by "continuous" we mean strongly continuous. An operator $T : X \supset D(T) \to X^*$ is demicontinuous at x_0 in $D(T)$ if it is continuous from the strong topology of $D(T)$ to the weak topology of X^*. It is "bounded" if it maps bounded subsets of $D(T)$ onto bounded sets. It is "compact" if it is continuous and maps bounded sets into relatively compact sets.

Definition 1. Let G be an open subset of X. An operator $C : \overline{G} \to X^*$ is said to be of type (S_+) if for every sequence $\{x_n\} \subset \overline{G}$ with $x_n \rightharpoonup x_0$ in X, and

$$\limsup_{n\to\infty} \langle Cx_n, x_n - x_0\rangle \le 0, \tag{1}$$

we have $x_n \to x_0 \in \overline{G}$.

The purpose of this paper is to show that the Nagumo approach may be employed in order to establish the Browder degree theory, for bounded demicontinuous (S_+) perturbations f of strongly quasibounded maximal monotone operators T (with $0 \in T(0)$) in separable spaces X. The resulting degree mapping $d(T+f, G, 0)$, where G is an open and bounded subset of X, is obtained via

$$d(T+f, G, 0) := \lim_{t\to 0} d_{NA}\left(I + \frac{1}{t}Q^*(T_t + f)Q, Q^{-1}G, 0\right), \tag{2}$$

where d_{NA} denotes the Nagumo degree for compact displacements of the identity in the Hilbert space H. This is a more direct approach than the Ibrahimou–Kartsatos one in [3], where one considers the degree mapping

$$d(T+f, G, 0) := \lim_{t\to 0} d_{LS}\left(I + \frac{1}{t}QQ^*(T_t + f), G, 0\right),$$

where d_{LS} denotes the Leray–Schauder degree mapping for the space X.

The most interesting feature of our new degree theory approach in this paper is that the approximating Nagumo degree mappings d_{NA} are for the embedded Hilbert space H, but not for the Banach space X.

The reason for the development of a degree theory in the present setting is that one might be able to use the Nagumo approach for the solvability of large classes of problems in Nonlinear Analysis by utilizing the compactness of the associated approximating operators in the way that has already been used in the past in the classical results of bifurcation, etc., for compact displacements of the identity. In addition to this fact, one might be able to solve, say, an existence problem in the Banach space X which is already known to hold for an analogous associated problem in the underlying separable Hilbert space H.

An application of elliptic super-regularization is also given involving ranges of sums of nonlinear operators via the use of the Skrypnik degree theory (cf. [4,5]) for demicontinuous, bounded and (S_+) mappings acting in the separable Hilbert space H.

For degree theories related to the content of this paper, the reader is referred to [6–12,3,13–16,4,5]. For some papers on degree theories and their applications to various problems in Nonlinear Analysis we cite [17–26]. For monotone operators and associated basic results, we cite [27,1,7,28,2,29,30]. The following fundamental result is due to Browder and Ton [31].

Theorem 1. *Let X be separable. Then there exists a real separable Hilbert space H and a linear compact injection $Q : H \to X$ such that $\overline{Q(H)} = X$.*

We identify H with H^* and let $Q^* : X^* \to H$ be the adjoint operator of Q. We observe that

$$\langle Q^*(w^*), v\rangle = \langle w^*, Q(v)\rangle, \quad (v, w^*) \in H \times X^*.$$

The operator Q^* is also compact and, since $Q(H)$ is dense in X, it is injective. In addition, since $Q : H \to X$ is continuous, we conclude that $Q^{-1}(\overline{G})$ is closed in H and $Q^{-1}(G)$ is open in H, for any open set $G \subset X$. Moreover,

$$\overline{Q^{-1}(G)} \subset Q^{-1}(\overline{G}), \quad \text{and} \quad \partial(Q^{-1}(G)) \subset Q^{-1}(\partial G). \tag{3}$$

A proof of the following lemma may be found in [24, Lemma 3.1] (cf. also [32, proof of Lemma 2.7]).

Lemma 1. *Let $T : X \supset D(T) \to 2^{X^*}$ be a maximal monotone operator with $0 \in D(T)$ and $0 \in T(0)$. Then the mapping $(t, x) \to T_t x$ is continuous on $(0, \infty) \times X$.*