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Global existence and uniform decay of a damped Klein–Gordon equation in a noncylindrical domain

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ABSTRACT

In this paper, we consider a damped Klein–Gordon equation in a noncylindrical domain. This work is devoted to proving the existence of global solutions and decay for the energy of solutions for a damped Klein–Gordon equation in a noncylindrical domain.

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1. Introduction

Let Ω be an open bounded domain of $\mathbb{R}^n$ containing the origin and having C^2 boundary. Let $\gamma : [0, \infty[\to \mathbb{R}$ be a continuously differentiable function. Consider the family of subdomains $\{\Omega_t\}_{0\le t<\infty}$ of $\mathbb{R}^n$ given by $\Omega_t = T(\Omega), T : y \in \Omega \mapsto x = \gamma(t)y$, whose boundaries are denoted by Γ_t, and let $\hat{Q}$ be the noncylindrical domain of $\mathbb{R}^{n+1}$ given by

$$\hat{Q} = \bigcup_{0\le t<\infty} \Omega_t \times \{t\}$$

with boundary

$$\hat{\Sigma} = \bigcup_{0\le t<\infty} \Gamma_t \times \{t\}.$$

In this paper, we are concerned with global existence and uniform decay of the energy to a damped Klein–Gordon equation given by

$$\begin{cases} u'' + au' - b\Delta u + k|u|^\rho u = f \quad \text{in } \hat{Q}, \\ u = 0 \quad \text{on } \hat{\Sigma}, \\ u(x, 0) = u_0, \qquad u'(x, 0) = u_1 \quad \text{in } \Omega_0, \end{cases} \tag{1.1}$$

where $b \ge 1$, a and k are positive constants and ρ is a nonnegative constant. Δ stands for the Laplacian with respect to the spatial variables; $'$ denotes the derivative with respect to time t.

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The Klein–Gordon equation is known as one of the nonlinear wave equations arising in relativistic quantum mechanics. For instance, Gao and Guo [1] studied the problem (1.1) with $\rho = 2$ and $n = 2$. They proved the existence and uniqueness of a time-periodic solution. Khalifa and Elgamal [2] proved local and global existence and uniqueness of the solution to (1.1) and they found numerical solutions to the one-dimensional Klein–Gordon equation.

The linear and nonlinear wave equations in noncylindrical domains have been treated by a number of authors. Lions [3] introduced the so-called penalty method for solving the problem of existence of solutions. Using this method, Cooper and Bardos [4] proved the existence and uniqueness of weak solutions of (1.1) with $a = 0$ and $b = k = 1$. Cavalcanti et al. [5] studied the existence and asymptotic behaviour of global regular solutions of the mixed problem for the Kirchhoff nonlinear model using a suitable change of variables. Ferreira and Lar'kin [6] studied existence and uniqueness of solutions for hyperbolic–parabolic equations in noncylindrical domains with the assumption $\sum_{i,j=1}^{n} \alpha_{ij}(y,t)\xi_i\xi_j \geq \beta|\xi|^2$, where β is a positive constant and α_{ij} is defined in $A(t)$. Thanks to this assumption, they were able to apply the Faedo–Galerkin method and prove existence and uniqueness of solutions. In this paper, we do not consider this assumption. In order to obtain the existence and uniqueness of solutions we use Gronwall's lemma, but we cannot apply Gronwall's lemma directly because of the term $A(t)v$. To overcome this problem we give a control of absolute value $\gamma'(t)$ (see (2.8)). We only obtained exponential decay of the solution for our problem for the case $n > 2$. This is because $E(t)$ is a nonincreasing function for the case $n > 2$, but for the case $n \leq 2$, $E(t)$ is not necessarily a decreasing function (see (4.3)). The main difficulty as regards obtaining the decay for the case $n \leq 2$ is due to the influence of the geometry of the noncylindrical domain and this is an open problem.

This paper is organized as follows: In Section 2, we recall the notation and hypotheses and introduce our main results. In Section 3, we prove the existence and uniqueness of strong solutions employing the Faedo–Galerkin method. In Section 4, we prove the exponential decay rate for the solution.

2. Hypotheses and main results

We begin this section by introducing some hypotheses and our main results. Throughout this paper we use standard functional spaces and define $\|\cdot\|_p$, $\|\cdot\|_{p,t}$ as the $L^p(\Omega)$ norm and $L^p(\Omega_t)$ norm. Also we define $(u,v) = \int_\Omega u(x)v(x)\mathrm{d}x$ and $(u,v)_t = \int_{\Omega_t} u(x)v(x)\mathrm{d}x$. And Δ and ∇ stand for the Laplacian and gradient with respect to the spatial variables respectively; $'$ denotes the derivative with respect to time t.

The method that we use to prove the result of existence and uniqueness is based on the transformation of our problem into another initial boundary value problem defined over a cylindrical domain whose sections are not time dependent. This is done using a suitable change of variables. Our existence result for the noncylindrical domain will follow using the inverse transformation—that is, using the diffeomorphism $\tau : \hat{Q} \to Q = \Omega \times \{t\}$ defined by

$$\tau : \hat{Q} \to Q, \qquad (x,t) \in \Omega_t \times \{t\} \mapsto (y,t) = \left(\frac{x}{\gamma(t)}, t\right) \tag{2.1}$$

and $\tau^{-1} : Q \to \hat{Q}$ defined by

$$\tau^{-1}(y,t) = (x,t) = (\gamma(t)y, t). \tag{2.2}$$

Denoting by v the function

$$v(y,t) = u \circ \tau^{-1}(y,t) = u(\gamma(t)y, t) \tag{2.3}$$

the initial–boundary value problem (1.1) becomes

$$\begin{cases} v'' + av' - b\gamma^{-2}\Delta v + k|v|^\rho v + A(t)v + \alpha_1 \cdot \nabla v' + \alpha_2 \cdot \nabla v = f & \text{in } Q, \\ v = 0 \quad \text{on } \Gamma, \\ v(y,0) = v_0, \qquad v'(y,0) = v_1 & \text{in } \Omega, \end{cases} \tag{2.4}$$

where

$$A(t)v = \sum_{i,j=1}^{n} \partial_{y_i}(\alpha_{ij}\partial_{y_j} v),$$
$$\alpha_{ij}(y,t) = (\gamma'\gamma^{-1})^2 y_i y_j,$$
$$\alpha_1(y,t) = -2\gamma'\gamma^{-1} y,$$
$$\alpha_2(y,t) = -\gamma^{-2}(\gamma''\gamma + \gamma'(a\gamma + (n-1)\gamma'))y.$$

The above method was introduced by Del Passo and Ughi [7] for studying a certain class of parabolic equations in noncylindrical domains.

Now we give the hypotheses for the main result.