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Stably average shadowable homoclinic classes

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ABSTRACT

Let p be a hyperbolic periodic saddle of a diffeomorphism of f on a closed smooth manifold M, and let $H_f(p)$ be the homoclinic class of f containing p. In this paper, we show that if $H_f(p)$ is locally maximal and every hyperbolic periodic point in $H_f(p)$ is uniformly far away from being nonhyperbolic, and $H_f(p)$ has the average shadowing property, then $H_f(p)$ is hyperbolic.

1. Introduction

It has been a main aim, as regards differentiable dynamical systems, for the last few decades, to understand the influence of a robust dynamic property on the behavior of the tangent map of the system. For instance, Mañé [3] proved that any robustly transitive diffeomorphism f of a closed surface S is an Anosov diffeomorphism. To study this problem, some people try to understand the influence of a robust dynamic property in systems with some shadowing properties, since these shadowing properties are closely related to the stability of systems (see [1,2,4–7]).

In this paper, we introduce the notion of the C^1-stably average shadowing property, and study the case where the homoclinic class has the stably average shadowing property.

Let us pass to the main definitions and results. Let M be a closed C^∞ Riemannian manifold. Denote by d the distance on M induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle TM. Let $\mathrm{Diff}(M)$ be the space of diffeomorphisms of M endowed with the C^1-topology. Let $f : M \to M$ be a diffeomorphism. For $\delta > 0$, a sequence of points $\{x_i\}_{i=a}^{b}$ $(-\infty \le a < b \le \infty)$ in M is called a δ-*pseudo-orbit* of f if

$$d(f(x_i), x_{i+1}) < \delta$$

for all $a \le i \le b-1$. For $\delta > 0$ a sequence $\{x_i\}_{i=-\infty}^{\infty}$ in M is called a δ-*average pseudo-orbit* of $f \in \mathrm{Diff}(M)$ if there is a natural number $N = N(\delta) > 0$ such that for all $n \ge N$, and $k \in \mathbb{Z}$,

$$\frac{1}{n}\sum_{i=1}^{n} d(f(x_{i+k}), x_{i+k+1}) < \delta.$$

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It is easy to see that a δ-pseudo-orbit is always a δ-average pseudo-orbit. We say that f has the average shadowing property or is *average shadowable* if for every $\epsilon > 0$ there is a $\delta > 0$ such that every δ-average pseudo-orbit $\{x_i\}_{i=-\infty}^{\infty}$ is ϵ-shadowed in average by some $z \in M$, that is,

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} d(f^i(z), x_i) < \epsilon.$$

Let $\Lambda \subset M$ be a closed f-invariant set. We say that $f|_\Lambda$ has the average shadowing property if for every $\epsilon > 0$ there is a $\delta > 0$ such that any δ-average pseudo-orbit $\{x_i\}_{i=a}^{b} \subset \Lambda$ of f is ϵ-shadowed in average by some $z \in \Lambda$.

Let $f \in \mathrm{Diff}(M)$, and let $\Lambda \subset M$ be a closed f-invariant set. We say that Λ is locally maximal if there is a compact neighborhood U of Λ such that

$$\bigcap_{n\in\mathbb{Z}} f^n(U) = \Lambda(U).$$

Note that f has the average shadowing property if and only if f^n has the average shadowing property for $n > 0$.

It is well known that if p is a hyperbolic periodic point f with period k then the sets

$$W^s(p) = \{x \in M : f^{kn}(x) \to p \text{ as } n \to \infty\}$$

and

$$W^u(p) = \{x \in M : f^{-kn}(x) \to p \text{ as } n \to \infty\}$$

are C^1-injectively immersed submanifolds of M.

Every point in the set $W^s(p)\overline{\pitchfork}W^u(p)$ is called a *homoclinic point* of f. The closure of the homoclinic points of f associated with p is called the *homoclinic class* of f and it is denoted by $H_f(p)$:

$$H_f(p) = \overline{W^s(p)\overline{\pitchfork}W^u(p)}.$$

Let $\Lambda \subset M$ be an f-invariant closed set. We say that Λ admits a *dominated splitting* if the tangent bundle $T_\Lambda M$ has a continuous Df-invariant splitting $T_\Lambda M = E^s \oplus E^u$, and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\left\|D_x f^n|_{E(x)}\right\| \cdot \left\|D_x f^{-n}|_{F(f^n(x))}\right\| \le C\lambda^n$$

for all $x \in \Lambda$ and $n \ge 0$.

Remark 1.1. An equivalent statement is the following: There are constants $m > 0$ and $\lambda \in (0, 1)$ such that

$$\|Df^m|_{E(x)}\|/m(Df^m|_{F(x)}) < \lambda^2,$$

for every $x \in \Lambda$.

We say that Λ is *hyperbolic* if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E^s \oplus E^u$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that $\|D_x f^n|_{E_x^s}\| \le C\lambda^n$ and $\|D_x f^{-n}|_{E_x^u}\| \le C\lambda^{-n}$ for all $x \in \Lambda$ and $n \ge 0$.

In the article, we prove the following conclusion:

Theorem A. *Suppose the homoclinic class $H_f(p)$ is locally maximal and satisfies the following properties:*

(a) *$H_f(p)$ admits a dominated splitting $T_{H_f(p)}M = E \oplus F$ with $\dim E = \dim W^s(p)$,*
(b) *there are constants $K > 0$, $m > 0$, $0 < \lambda < 1$ such that if $q \in H_f(p)$ has minimum period $\pi(q) \ge m$, then*

$$\prod_{i=0}^{k-1} \|D_{f^{im}(q)} f^m|_{E^s|_{f^{im}(q)}}\| < K\lambda^k \quad \text{and} \quad \prod_{i=0}^{k-1} \|D_{f^{-im}(q)} f^m|_{E^u|_{f^{-im}(q)}}\| < K\lambda^k,$$

where $k = [\pi(q)/m]$,
(c) *f has the average shadowing property on $H_f(p)$;*

then $H_f(p)$ is hyperbolic for f.

Actually, the above theorem comes from the consideration of the so-called C^1-stably average shadowable homoclinic class.

Definition 1.2. Let Λ be a closed f-invariant set. We say that an $f \in \mathrm{Diff}(M)$ has the C^1-*stably average shadowing property on* Λ or Λ is C^1-*stably average shadowable* if there exist a C^1-neighborhood $\mathcal{U}(f)$ of f and a compact neighborhood U of Λ such that:

(i) $\Lambda(U) = \bigcap_{n\in\mathbb{Z}} f^n(U)$, i.e., Λ is locally maximal;
(ii) for any $g \in \mathcal{U}(f)$, $g|_{\Lambda_g(U)}$ has the average shadowing property, where $\Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U)$, which is called the *continuation* of $\Lambda(U)$.