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Integrability for solutions to some anisotropic elliptic equations

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ABSTRACT

We consider the boundary value problem

$$\begin{cases} \sum_{i=1}^{n} D_i(a_i(x, Du(x))) = 0, & x \in \Omega; \\ u(x) = u_*(x), & x \in \partial\Omega. \end{cases}$$

We show that, higher integrability of the boundary datum u_* forces solutions u to have higher integrability as well. Assumptions on $a_i(x, z)$ are suggested by the Euler equation of the anisotropic functional

$$\int_{\Omega} (|D_1 u|^{p_1} + |D_2 u|^{p_2} + \cdots + |D_n u|^{p_n})dx.$$

1. Introduction

We consider integral functionals

$$\mathcal{I}(u) = \int_{\Omega} f(x, Du(x))dx \tag{1.1}$$

where $u: \Omega \subset \mathbb{R}^n \to \mathbb{R}$ and $f: \Omega \times \mathbb{R}^n \to [0, +\infty)$; about $f(x, z)$, we assume that $x \to f(x, z)$ is measurable and $z \to f(x, z)$ is continuous; u is taken from the Sobolev space $W^{1,1}(\Omega)$. We are interested in functions u minimizing $\mathcal{I}$ or solving its Euler equation

$$\sum_{i=1}^{n} D_i\left(\frac{\partial f}{\partial z_i}(x, Du(x))\right) = 0 \tag{1.2}$$

in weak form, or more generally

$$\sum_{i=1}^{n} D_i(a_i(x, Du(x))) = 0. \tag{1.3}$$

In past years, great attention has been paid to anisotropic functionals whose model is

$$\int_{\Omega} (|D_1 u|^{p_1} + |D_2 u|^{p_2} + \cdots + |D_n u|^{p_n})dx \tag{1.4}$$

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where the derivative $D_i u = \frac{\partial u}{\partial x_i}$ has the exponent p_i that might be different from the exponent p_j of the derivative $D_j u = \frac{\partial u}{\partial x_j}$, when $j \neq i$. Such a model suggests to consider energies $f(x, z)$, where

$$\sum_{i=1}^{n} |z_i|^{p_i} \le f(x, z) \le c \left(1 + \sum_{i=1}^{n} |z_i|^{p_i} \right) \tag{1.5}$$

or Eqs. (1.3) with coefficients $a_i(x, z)$ satisfying

$$|a_i(x, z)| \le c(1 + |z_i|)^{p_i - 1}. \tag{1.6}$$

This anisotropic framework looks useful when dealing with some reinforced materials; see [1]; about theoretical viewpoint see [2, example 1.7.1, page 169]. A fundamental result in the regularity theory for minimizers in the anisotropic setting is contained in [3], where the Lipschitz continuity is proved. As far as fractional differentiability is concerned, [4] shows that the boundedness of minimizers is an important tool; see the estimate after formula (4.15) in [4]; in order to prove boundedness, one could use the maximum principle; see Theorem 3.3, Chapter 5 in [5] and [6]. A smart remark [7] about the proof given in [4] suggests that boundedness of minimizers is not needed: only a high degree of integrability for minimizers is required. The aim of the present paper is to show that higher integrability of the boundary datum u_* forces solutions u to have higher integrability as well. Precise assumptions and the statement are given in the next section. Here we want to make a few remarks about the proof. When showing boundedness of u, we usually "cut" u at some level $L \ge \sup_{\partial\Omega} u$ in such a way that $\min\{u; L\}$ has the same boundary values as u; then $u - \min\{u; L\}$ vanishes on the boundary of Ω and we test Eq. (1.3) with such a function: we get information on the measure of the superlevel $\{x \in \Omega : u(x) > L\}$. When the boundary datum is no longer bounded, $\sup_{\partial\Omega} u$ might be infinity and such a "cut" is no longer allowed. In the present paper, we consider the difference $u - u_*$ between the solution u and the boundary datum u_*: such a difference vanishes on the boundary of Ω; then we "cut" such a difference, test the equation and get information on the measure of $\{x \in \Omega : u(x) - u_*(x) > L\}$. Since $u = u_* + (u - u_*)$, we arrive at higher integrability of u. In Section 2, we write precise assumptions and the statement, whose proof appears in Section 3. We end this introduction by remarking that this paper is concerned with higher integrability of u; as far as higher integrability of Du is concerned, a delicate interplay between the regularity of $x \to f(x, z)$ and the growth of $z \to f(x, z)$ appears: see [8].

2. Assumptions and results

Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \ge 2$. Let $a_i : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be Carathéodory functions, that is, $a_i(x, z)$ are measurable with respect to x and continuous with respect to z. We assume anisotropic growth: there exist $p_1, \ldots, p_n \in (1, +\infty)$ and $c_2 \in (0, +\infty)$ such that

$$|a_i(x, z)| \le c_2(1 + |z_i|)^{p_i - 1} \tag{2.1}$$

for almost every $x \in \Omega$, for every $z \in \mathbb{R}^n$, and for any $i = 1, \ldots, n$. Moreover, we suppose that the following anisotropic monotonicity condition holds. There exists $\tilde{\nu} \in (0, +\infty)$ such that

$$\tilde{\nu} \sum_{i=1}^{n} |z - \tilde{z}|^{p_i} \le \sum_{i=1}^{n} (a_i(x, z) - a_i(x, \tilde{z}))(z_i - \tilde{z}_i) \tag{2.2}$$

for almost every $x \in \Omega$, for any $z, \tilde{z} \in \mathbb{R}^n$. We introduce the anisotropic Sobolev space:

$$W_0^{1,(p_i)}(\Omega) = \{v \in W_0^{1,1}(\Omega) : D_i v \in L^{p_i}(\Omega) \text{ for every } i = 1, \ldots, n\}.$$

For the boundary datum $u_* : \Omega \to \mathbb{R}$, we assume that:

$$u_* \in W^{1,1}(\Omega) \quad \text{with } D_i u_* \in L^{q_i}(\Omega) \text{ and } q_i \in (p_i, +\infty) \tag{2.3}$$

for every $i = 1, \ldots, n$. Let $\bar{p}$ be the harmonic mean of p_i, i.e.:

$$\frac{1}{\bar{p}} = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{p_i}; \tag{2.4}$$

by assuming that $\bar{p} < n$, we can introduce the Sobolev conjugate $\bar{p}^* = \frac{n\bar{p}}{n - \bar{p}}$. Our next goal is to prove the following

Theorem 2.1. *Under previous assumptions, (2.1)–(2.3), let $u \in u_* + W_0^{1,(p_i)}(\Omega)$ verify*

$$\int_{\Omega} \sum_{i=1}^{n} a_i(x, Du(x)) D_i v(x) dx = 0 \quad \forall v \in W_0^{1,(p_i)}(\Omega). \tag{2.5}$$