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Positive solutions for fourth-order differential equations with deviating arguments and integral boundary conditions

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ABSTRACT

In this paper we investigate integral boundary value problems for fourth order differential equations with deviating arguments. We discuss our problem both for advanced or delayed arguments. We establish sufficient conditions under which such problems have positive solutions. To obtain the existence of multiple (at least three) positive solutions, we use a fixed point theorem due to Avery and Peterson. An example is also included to illustrate that corresponding assumptions are satisfied. The results are new.

1. Introduction

Let $J = [0, 1], J_0 = (0, 1), \mathbb{R}_+ = [0, \infty)$. In this paper, we shall consider the existence of multiple positive solutions for a class of fourth-order differential equations of type

$$x^{(4)}(t) = h(t)f(t, x(t), x(\alpha(t))), \quad t \in J_0 \tag{1}$$

subject to the boundary conditions

$$\begin{cases} x(0) = \gamma x'(0) - \int_0^1 g(s)x(s)ds, \\ x(1) = \beta x(\eta), \qquad x''(0) = x''(1) = 0 \end{cases} \tag{2}$$

or

$$\begin{cases} x(0) = \beta x(\eta), \qquad x''(0) = x''(1) = 0, \\ x(1) = \gamma x'(0) - \int_0^1 g(s)x(s)ds. \end{cases} \tag{3}$$

Note that x in formula (1) depends on argument α which can be both of advanced or delayed type.

We introduce the following assumptions:

(H_1): $f \in C(J \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+), g \in C(J, \mathbb{R}_+), \alpha \in C(J, J)$,
(H_2): h is a nonnegative continuous function defined on $(0, 1)$; h is not identically zero on any subinterval on $(0, 1)$.

We have many fixed point theorems including corresponding theorems in a cone. Recently, many authors have been interested in studying the existence of positive solutions for differential equations with boundary conditions. There exists a vast amount of literature devoted to the applications of fixed point theorems concerning positive solutions of boundary

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value problems to second-order differential equations, see for example [1–11] and for fourth-order differential equations, see for example [12–19]; see also the references therein. Boundary value problems with integral conditions constitute an important class of such problems, see for example [3,5]. A fixed point theorem in a cone can also be applied to differential equations with deviating arguments but there are only a few papers where such techniques are applied, see for example [6,7,11]. Indeed, papers [1,2,4,8–10] have also a word "delay" in the title but the corresponding functions f appearing on the right-hand-side depend on $x(t-\tau),\ \ \tau>0$, where initial functions x are given on the initial set, for example $[-\tau,0]$. In such cases $\alpha(t)=t-\tau$, so we have problems with a constant delay τ. If we consider the differential problem on intervals $[0,k]$, where $k\le\tau$, then it means that we have no delays; we have such a situation in paper [10]. If $k>\tau$, then it is easy to solve the differential equation on $[0,\tau]$, since we have the solution on the initial set $[-\tau,0]$. Continuing this process, we can find a solution on the whole interval $[0,k]$, by using the method of steps. In my paper, for example, the deviating argument α can have a form $\alpha(t)=\rho t=t-(1-\rho)t$ with a fixed number $\rho\in(0,1)$, so the delay $(1-\rho)t$ is a function of t. In this case, the initial set reduces to one point $t=0$, and we can not apply the step method. To my knowledge, it is the first paper when positive solutions have been investigated for the integral boundary conditions of fourth-order differential equations with deviating arguments both of an advanced or delayed type.

The organization of this paper is as follows. In Section 2, we present some necessary lemmas connected with the case when problem (1)–(2) is of advanced type. In Section 3, we present some definitions and a theorem of Avery and Peterson which are useful to obtain our main results. In Section 4, we discuss the existence of at least three positive solutions of problem (1)–(2) with advanced argument α by using the Avery–Peterson theorem. At the end of this section, an example is added to verify theoretical results. In Section 5, we formulate sufficient conditions under which delayed problem (1), (3) has at least three positive solutions.

2. Some lemmas

Let us consider the following problem

$$\begin{cases}u^{(4)}(t)+y(t)=0, & t\in J_0,\\ u''(0)=u''(1)=0,\end{cases}\tag{4}$$

$$u(0)=\gamma u'(0)-\int_0^1 g(s)u(s)\mathrm{d}s,\tag{5}$$

$$u(1)=\beta u(\eta).\tag{6}$$

Put

$$g_1=\int_0^1 g(s)\mathrm{d}s,\qquad g_2=\int_0^1 sg(s)\mathrm{d}s,$$

$$m(t)=t\int_0^1(1-s)y(s)\mathrm{d}s-\int_0^t(t-s)y(s)\mathrm{d}s,$$

$$\Delta=(1+g_1)(1-\beta\eta)+(\gamma-g_2)(1-\beta),\qquad G=\int_0^1 g(s)\int_0^s(s-\tau)m(\tau)\mathrm{d}\tau\mathrm{d}s.$$

We assume that:

(H_3): $\eta\in(0,1),0<\beta<\frac{1}{\eta},g_1\ge 0,\gamma\ge g_2$ and $\Delta>0$.

We require the following

Lemma 1. *Assume that $\eta\in(0,1),\ \ \Delta\ne 0$ and $y\in C(J,\mathbb{R})$. Then problem (4)–(6) has the unique solution given by the following formula*

$$u(t)=\frac{1}{\Delta}\left\{[\gamma-g_2+t(1+g_1)]\left[\beta\int_0^\eta(\eta-s)m(s)\mathrm{d}s-\int_0^1(1-s)m(s)\mathrm{d}s\right]\right.$$
$$\left.-[1-\beta\eta-t(1-\beta)]G\right\}+\int_0^t(t-s)m(s)\mathrm{d}s,\quad t\in J.\tag{7}$$

Proof. Integrating two times the differential equation in (4) and using the boundary conditions in (4) we have

$$u''(t)=m(t).\tag{8}$$

Next integrating two times Eq. (8) we have

$$u(t)=u(0)+u'(0)t+\int_0^t(t-s)m(s)\mathrm{d}s.\tag{9}$$