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Spatial and temporal decay estimate of the Stokes flow of weighted L^1 initial data in the half space☆

Bum Ja Jin

Department of Mathematics, Mokpo National University, Muan 534-729, South Korea

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ABSTRACT

In this paper, we derive the spatial and temporal decay estimate of the Stokes solution $e^{-tA}\mathbf{a}$ in the half space when the prescribed initial data is in a weighted L^1 space. We obtained that

$$\|x_i^r e^{-tA}\mathbf{a}\|_{L^\infty} \le Ct^{-\frac{n}{2}-\frac{1}{2}}\|x_i^r x_n \mathbf{a}\|_{L^1} + Ct^{-\frac{n}{2}+\frac{r-1}{2}}\|x_n\mathbf{a}\|_{L^1}$$

for $0 \le r < n$ if $i = n$ and $0 \le r < n-1$ if $i = 1, \ldots, n-1$.

1. Introduction

Let us consider the Stokes equations in the n-dimensional half space $\mathbb{R}^n_+$, $n \ge 2$:

$$\operatorname{div} \mathbf{u} = 0, \quad \frac{\partial \mathbf{u}}{\partial t} - \Delta \mathbf{u} + \nabla p = 0 \text{ in } \mathbb{R}^n_+ \times (0,\infty) \tag{1}$$

with initial condition $\mathbf{u}(\mathbf{x}, 0) = \mathbf{a}(\mathbf{x})$ in $\mathbb{R}^n_+$ and with no slip boundary condition $\mathbf{u} = 0$ on $x_n = 0$. Here $\mathbf{a}$ is a divergence free vector field with zero boundary value.

Let A be the generating operator of the Stokes semi-group so that the solution of the Stokes equation (1) is denoted by $\mathbf{u}(\mathbf{x}, t) = e^{-tA}\mathbf{a}(\mathbf{x})$.

The $L^q - L^1$ estimate of the Stokes flow in the half space has been derived by Borchers and Miyakawa [1] via semi-group operator theory for $1 < q \le \infty$:

$$\|e^{-tA}\mathbf{a}\|_{L^q} \le Ct^{-\frac{n}{2}(1-\frac{1}{q})}\|\mathbf{a}\|_{L^1}.$$

Desch, Hieber and Pruss [2] derived $L^\infty - L^\infty$ estimate and showed an example that the $L^1 - L^1$ estimate does not hold.

The $L^1 - L^1$ estimate for $\nabla \mathbf{u}$ in the half space has been shown by Giga, Matsui and Shimizu [3] for $q = 1$; by Shimizu [4] for $q = \infty$; by Shibara and Shimizu [5] for $1 \le q \le \infty$.

Fujigaki and Miyakawa [6] derived more rapid L^q estimate of the Stokes flow with initial data in the weighted L^1 space:

$$\|e^{-tA}\mathbf{a}\|_{L^q} \le Ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})}\|x_n\mathbf{a}\|_{L^1}.$$

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E-mail address: bumjajin@hanmail.net.

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Bae [7] has considered a rapid L^1 estimate and a rapid L^∞ estimate with an initial data with the special condition $\int_{-\infty}^{\infty} \mathbf{a}(\mathbf{x})d\mathbf{x}_i = 0$ for some $i = 1, \ldots, n-1$, and in this case

$$\|e^{-tA}\mathbf{a}\|_{L^p} \le Ct^{-\frac{1}{2}}\|x_n\mathbf{a}\|_{L^p}, \quad p = 1, \infty.$$

When a spatially decaying initial data is given, Crispo and Maremonti [8] derived a spatial and temporal decay estimate of the Stokes flow. Making use of it, they proved the existence of spatially decaying Navier–Stokes flow.

Bae [9] derived weighted L^q estimates of the Stokes flow for $1 < q < \infty$ when the prescribed initial data is in weighted L^1 space. (See also the paper of Jin [10] for more sophisticated estimate.)

In this paper, we derive the spatial and temporal decay estimate of the Stokes flow when the prescribed initial data is in weighted L^1 space. The following is the statements of the main theorem. (In other words, we consider the weighted L^∞ estimate for weighted L^1 data.)

Theorem 1.1. *Let $x_n(1+|x_i|)^r\mathbf{a} \in L^1$ for $0 \le r < n-1$ and $i = 1, \ldots, n-1$ and let $x_n(1+|x_n|)^r\mathbf{a} \in L^1$ for $0 \le r < n$. Then we have*

$$\|x_i^r e^{-tA}\mathbf{a}\|_{L^\infty(\mathbb{R}^n_+)} \le C[t^{-\frac{n}{2}-\frac{1}{2}}\|x_n x_i^r\mathbf{a}\|_{L^1(\mathbb{R}^n_+)} + t^{-\frac{n}{2}+\frac{r-1}{2}}\|x_n\mathbf{a}\|_{L^1(\mathbb{R}^n_+)}] \tag{2}$$

for $0 \le r < n-1$ if $i = 1, \ldots, n-1$ and

$$\|x_n^r e^{-tA}\mathbf{a}\|_{L^\infty(\mathbb{R}^n_+)} \le C[t^{-\frac{n}{2}-\frac{1}{2}}\|x_n^{r+1}\mathbf{a}\|_{L^1(\mathbb{R}^n_+)} + t^{-\frac{n}{2}+\frac{r-1}{2}}\|x_n\mathbf{a}\|_{L^1(\mathbb{R}^n_+)}] \tag{3}$$

for $0 \le r < n$.

The above theorem comes from the following estimates.

Theorem 1.2. *Let $x_n(1+|x_i|)^r\mathbf{a} \in L^1$ for $0 \le r < n$, $i = 1, \ldots, n$. Then we have*

$$\|x_i^r[e^{-tA}\mathbf{a} - e^{-tB}(\bar{\mathbf{a}} + Sa_n)]\|_{L^\infty(\mathbb{R}^n_+)} \le C[t^{-\frac{n}{2}-\frac{1}{2}}\|x_n x_i^r\mathbf{a}\|_{L^1(\mathbb{R}^n_+)} + t^{-\frac{n}{2}+\frac{r-1}{2}}\|x_n\mathbf{a}\|_{L^1(\mathbb{R}^n_+)}].$$

Here $\mathbf{a} = (\bar{\mathbf{a}}, a_n)$, and e^{-tB} and S are defined in Section 2.

Theorem 1.2 is obtained by combining Theorem 4.2 in Section 4, Theorem 5.2 in Section 5, and Theorem 6.2 in Section 6.
Theorem 1.1 is obtained by combining Theorem 1.2 for the estimate of $e^{-tA}\mathbf{a} - e^{-tB}(\bar{\mathbf{a}} + Sa_n)$, Lemma 3.2 for the estimate of $e^{-tB}\bar{\mathbf{a}}$ and Lemma 3.3 for the estimate of $e^{-tB}Sa_n$.

Remark 1.1. The proof of Theorem 1.1 for $r = 0$ has been already done in [6]. They made use of $L^q - L^1$ estimate in [1] which is obtained via semi-group operator theory. In this paper, we derives it by direct computation via Fourier transform.

Remark 1.2. We compare the result of [8] with the result of this paper.

- In [8], a spatial and temporal estimate of the Stokes flow has been derived for a rapidly decaying smooth initial data satisfying $(1+|\mathbf{x}|)^r\mathbf{a} \in L^\infty(\mathbb{R}^n_+)$ for $0 < r < n$.
  Moreover, it is shown that
  $$t^{\frac{\alpha}{2}}\|(1+|\mathbf{x}|)^\beta e^{-tA}\mathbf{a}\|_{L^\infty(\mathbb{R}^n_+)} \le C\|(1+|\mathbf{x}|)^r\mathbf{a}\|_{L^\infty(\mathbb{R}^n_+)}, \qquad \alpha + \beta = r < n.$$
  Hence the spatial decay rate is less than n.
- In this paper, our initial data $\mathbf{a}$ may have a singularities, since we assume $(1+|x_i|)^r x_n\mathbf{a} \in L^1(\mathbb{R}^n_+)$.
  In our result, the spatial decay rate in tangential direction is less than $n-1$. By our technique $n-1$ seems to be optimal for the tangential decay rate when the initial data is in the weighted L^1space. However, it should be clarified and we leave it for the future.

There are recent papers on Stokes equations and Navier–Stokes equations in the half space with nondecaying initial data. Maremonti and Starita [11] derived the estimates of the Stokes flow with bounded or continuous initial velocity with $(1+|\mathbf{x}|^\beta)\mathbf{a} \in L^\infty(\mathbb{R}^n_+)$, $0 \le \beta < 1$. Making use of it, Maremonti [12] proved the existence (local in time) and uniqueness of classical solutions to the Navier–Stokes equations under the condition that initial data are only continuous and bounded.

Remark 1.3. Solution formula of the Stokes equations in the half space have been derived by Solonnikov [13] and Ukai [14], independently.

In [13], Solonnikov's solution formula is represented by Green functions and it has been used mainly in L^∞ framework. (See [8,12,11,15].)

In [14], Ukai's solution formula is represented by the compositions of n-dimensional Riesz operators, $(n-1)$-dimensional Riesz operators, and solution operators of heat equations, and it has been used in the L^q framework, mainly for $1 < q < \infty$. (See [7,9,1,6,3,5,4].)