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Quasi-linear boundary value problems with generalized nonlocal boundary conditions

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ABSTRACT

We investigate a quasi-linear boundary value problem of the form $-\mathrm{div}(\alpha|\nabla u|^{p-2}\nabla u) = 0$ involving a general boundary map and mixed Neumann boundary conditions on a bounded Lipschitz domain. We show existence, uniqueness, and Hölder continuity of the weak solution of this mixed boundary value problem, and obtain maximum principles for this class of mixed equations. As a consequence, we obtain uniform continuity up to the boundary to solutions associated with a class of electrical models described by Maxwell's equations with nonlocal boundary conditions. An extension to boundary value problems with generalized nonlocal Robin boundary conditions is also achieved.

1. Introduction

Let $p \in (1, \infty)$ and let $\Omega \subseteq \mathbb{R}^N$ be a bounded Lipschitz domain. Let Γ_η and Γ_ς be a decomposition of the boundary $\partial\Omega$, in the sense that $\Gamma_\eta \cup \Gamma_\varsigma = \partial\Omega$ and $\Gamma_\eta \cap \Gamma_\varsigma = \emptyset$. The aim of this article is to investigate the quasi-linear mixed boundary value problem formally given by

$$\begin{cases} -\mathrm{div}(\alpha|\nabla u|^{p-2})\nabla u = \Phi & \text{in } \Omega, \\ \alpha|\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \Lambda_p u & \text{on } \Gamma_\eta, \\ \alpha|\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \Psi & \text{on } \Gamma_\varsigma. \end{cases} \tag{1.1}$$

The coefficient α is assumed to be a bounded strictly positive measurable function over $\overline{\Omega}$, Φ and Ψ belong to $(W^{1,p}(\Omega))^*$ and $\mathfrak{B}^p_{1-1/p}(\partial\Omega, \mathrm{d}\sigma)^*$, respectively (see the next section for the definition of these function spaces), and Λ_p denotes a general boundary map from $\mathfrak{B}^p_{1-1/p}(\partial\Omega, \mathrm{d}\sigma)$ to $\mathfrak{B}^p_{1-1/p}(\partial\Omega, \mathrm{d}\sigma)^*$ (see Section 3 for more details). Here σ denotes the restriction

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to $\partial\Omega$ of the $(N-1)$-dimensional Hausdorff measure. Observe that Eq. (1.1) corresponds to a mixed Neumann problem, although we will also make the application to a generalized Robin problem, formally defined by

$$\begin{cases} -\mathrm{div}(\alpha|\nabla u|^{p-2})\nabla u + \lambda|u|^{p-2}u = \Phi & \text{in } \Omega, \\ \alpha|\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} + \beta_\eta |u|^{p-2}u = \Lambda_p u & \text{on } \Gamma_\eta, \\ \alpha|\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} + \beta_\varsigma |u|^{p-2}u = \Psi & \text{on } \Gamma_\varsigma, \end{cases} \tag{1.2}$$

for nonnegative coefficients $\lambda \in L^\infty(\Omega, \mathrm{d}x)$, $\beta_\eta \in L^\infty(\Gamma_\eta, \mathrm{d}\sigma)$, and $\beta_\varsigma \in L^\infty(\Gamma_\varsigma, \mathrm{d}\sigma)$. The major innovation of the above equations is given by the general boundary conditions involving the operator Λ_p, which in many cases can be of nonlocal nature.

The classical Neumann boundary value problem has been intensely investigated by many authors, and in particular, global regularity of weak solutions is well known (e.g. [1–3], and the references therein). Similar results have been obtained for local Robin problems (e.g. [3,4], among many others). The crucial achievement was the work of Nittka [3], who obtained Hölder continuity up to the boundary of weak solutions for the p-Laplacian with local Neumann and Robin boundary conditions. Recently, an equation with nonlocal Robin boundary conditions has been studied in [5–7], and in particular the author obtained in [5], the same regularity results for weak solutions as in [3], but for the linear case and nonlocal boundary conditions given by a specific boundary map (see Section 6). In this paper, we will generalize the results of [3,5] discussed above, by obtaining the same regularity results for Eqs. (1.1) and (1.2). In particular, our selection of the boundary operator, Λ_p, will include the nonlocal map given in [7,5] for $p = 2$.

The framework of Eq. (1.1) was motivated in some part by the work of Jonsson [8], who considered the linear version of Eq. (1.1) for a general boundary map Λ_2^γ with similar properties as the partial Dirichlet-to-Neumann map (e.g. [8,9]). We will show that our choice of the boundary map Λ_p will include the operator considered by Jonsson on [8] for $p = 2$. Consequently, we will derive regularity results for solutions associated with the equation considered by Jonsson in [8].

We organize the paper as follows. In Section 2 we provide some basic definitions and properties of functions spaces, and state some known results that will be applied throughout the rest of the article. In Section 3 we pose the mixed problem and impose the preliminary assumptions, namely, some basic properties for α, Φ, and Ψ, as well as some monotonicity and boundedness properties for the boundary map Λ_p. Then we show that the problem (1.1) admits a unique weak solution $u \in W^{1,p}(\Omega)$ with zero mean, in the sense that

$$\int_\Omega \alpha(x)|\nabla u|^{p-2}\nabla u \nabla\varphi \,\mathrm{d}x - (\Lambda_p u)(\varphi) = \Phi(\varphi) + \Psi(\varphi) \quad \forall \varphi \in W^{1,p}(\Omega).$$

Section 4 features a priori estimates of weak solutions of (1.1). To be more precise, assuming that Φ, Ψ are L^q and L^r-functions for appropriate values $q, r \in (1, \infty]$, and under a further condition on the operator Λ_p, we establish global boundedness of weak solutions of (1.1), and give a relation between such solutions and the boundary map Λ_p, similar to the one given in [8]. In Section 5 we state and prove the main result of this paper, namely, the Hölder continuity over $\overline{\Omega}$ of weak solutions of (1.1). In addition, some maximum principles are derived for the problem (1.1) under some extra assumption on the map Λ_p. Finally, Section 6 presents examples of boundary maps satisfying the conditions prescribed in the previous sections, and discuss an application to the work of Jonsson [8] related to Maxwell's equation. Also, an extension to the generalized mixed Robin boundary value problem (1.2) is given, were we obtain the same conclusions as before for this problem. At the end, we give an innovate application to boundary value problems of the form (1.1) and (1.2) with nonlocal boundary data, where we obtain the same conclusions previously described for this situation.

2. Notation and preliminaries

Throughout this paper we will consider $\Omega \subseteq \mathbb{R}^N$ as a bounded Lipschitz domain. We denote by $|\cdot|$ the usual N-dimensional Lebesgue measure on Ω, and σ the restriction to $\partial\Omega$ of the $(N-1)$-dimensional Hausdorff measure, which in this case coincides with the usual Lebesgue surface measure on $\partial\Omega$. We separate the boundary $\partial\Omega$ of Ω into two disjoint boundary parts Γ_η, and $\Gamma_\varsigma := \partial\Omega \setminus \Gamma_\eta$.

As usual, by $W^{1,p}(\Omega)$ we mean the L^p-based Sobolev space, and we denote by $\mathfrak{B}^p_{1-1/p}(\partial\Omega, \mathrm{d}\sigma)$ the classical Besov space on $\partial\Omega$ relative to the measure σ, which is the set of functions $u \in L^p(\partial\Omega, \mathrm{d}\sigma)$ such that the semi-norm

$$\mathcal{N}^p_{1-1/p}(u, \partial\Omega, \mathrm{d}\sigma) := \int_{\partial\Omega}\int_{\partial\Omega}\left(\frac{|u(x)-u(y)|}{|x-y|^{1-1/p}}\right)^p \frac{1}{|x-y|^{N-1}} \mathrm{d}\sigma_x \mathrm{d}\sigma_y$$

is finite. It is well known that $\mathfrak{B}^p_{1-1/p}(\partial\Omega, \mathrm{d}\sigma)$ is a Banach space under the norm

$$\|\cdot\|^p_{\mathfrak{B}^p(\partial\Omega)} := \|\cdot\|^p_{\mathfrak{B}^p_{1-1/p}(\partial\Omega,\mathrm{d}\sigma)} := \|\cdot\|^p_{p,\partial\Omega} + \mathcal{N}^p_{1-1/p}(\cdot, \partial\Omega, \mathrm{d}\sigma),$$

which is reflexive if $p \in (1, \infty)$. Recall that for a Banach space E, its dual space will be denote by E^*. We collect the following well-known Sobolev and Besov embedding and trace properties (e.g. [10–19], and the references therein), since most of them will be frequently applied in the subsequent sections.