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A Neumann problem for the KdV equation with Landau damping on a half-line

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ABSTRACT

We consider the initial-boundary value problem on a half-line for the KdV equation with Landau damping. We study traditionally important problems of the theory of nonlinear partial differential equations, such as global in time existence of solutions to the initial-boundary value problem and the asymptotic behavior of solutions for large time.

1. Introduction

We consider the initial-boundary value problem on a half-line for the KdV equation with Landau damping:

$$\begin{cases} u_t + uu_x - |\alpha| u_{xxx} + \mathcal{R}^{\frac{1}{2}} \partial_x u = 0, & t > 0,\ x > 0, \\ u(x, 0) = u_0(x), & x > 0, \\ u_x(0, t) = u_{xx}(0, t) = 0, & t > 0, \end{cases} \tag{1.1}$$

where $\alpha \in \mathbb{R}$, and

$$\mathcal{R}^{\beta} \phi = \frac{1}{2\Gamma(\beta) \sin\left(\frac{\pi}{2}\beta\right)} \int_0^{+\infty} \frac{\phi(y)}{|x-y|^{1-\beta}} \mathrm{d}y$$

is the modified Riesz potential (see [1], p. 214).

Due to the intensive development of the theory and applications, the initial-boundary value problem (1.1) plays an important role in modern science. Apart from diverse areas of mathematics, nonlocal partial differential evolution equations arise in modern mathematical physics and many other branches of science, such as, for example, chemical physics and electrical networks (for details, see [2–9]).

Many articles have appeared in the literature, where fractional derivatives are used for a better description of certain material properties. For example, Ott et al. [10–12] proposed the following generalizations of the KdV equation

$$u_t + uu_x + \alpha u_{xxx} + \mathcal{R}^{\frac{1}{2}} \partial_x u = 0. \tag{1.2}$$

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This equation is a simple universal model describing the ion-acoustic waves in a plasma with Landau damping. It combines the shallow water equation with the fractional derivative term that produces a wave damping. The Cauchy problems for some classes of nonlinear equations with fractional derivative were studied intensively (for details see, for example, [13] and its references). The applied problem naturally leads to the necessity of the formulation of the initial-boundary value problem to this equation. However their mathematical investigation is more complicated comparing with the Cauchy problems (see [14,15] and references therein). To prove the well-posedness of the problem we need to solve a question of how many boundary values should be given in the problem for its solvability and the uniqueness of the solution. Also it is important to study the influence of the boundary data on the qualitative properties of the solution. It should be noted that most papers and books on fractional calculus are devoted to solvability of fractional ordinary differential equations (ODE). However, few works have considered the initial-boundary value problems for partial differential evolution equations with a fractional derivative.

For the general theory of nonlinear equations with a fractional derivative on a half-line we refer to the book [16]. This book is the first attempt to develop systematically a general theory of the initial-boundary value problems for evolution equations with pseudo-differential operators on a half-line. The pseudo-differential operator $\mathbb{K}$ on a half-line was introduced by virtue of the inverse Laplace transformation of the product of the symbol $K(p) = O(p^\alpha)$ which is analytic in the right complex half-plane, and the Laplace transform of the derivative $\partial_x^{[\alpha]}u$. Thus, for example, in the case of $K(p) = p^\alpha$ we get the following definition of the fractional derivative ∂_x^α,

$$\partial_x^\alpha = \mathcal{L}^{-1}\left\{p^\alpha\left(\mathcal{L} - \sum_{j=1}^{[\alpha]}\frac{\lim_{x\to 0^+}\partial_x^{j-1}}{p^j}\right)\right\}. \tag{1.3}$$

Here and below p^α is the main branch of the complex analytic function in the complex half-plane Re $p \geq 0$, so that $1^\alpha = 1$ (we make a cut along the negative real axis $(-\infty, 0)$). Note that due to the analyticity of p^α for all Re $p > 0$ the inverse Laplace transform gives us the function which is equal to 0 for all $x < 0$. In order to obtain an explicit form of the Green function it was used an approach based on the Laplace transformation with respect to the spatial variable instead of the standard application of the Laplace transformation with respect to the time variable. Methods of this book can be applied directly to study the initial-boundary value problem for differential equations with fractional Riemann–Liouville derivative

$$\partial_x^\alpha = \frac{1}{\Gamma(\alpha - [\alpha])}\int_0^x \frac{\partial_y^{[\alpha]+1}}{(x-y)^{\alpha-[\alpha]}}\mathrm{d}y.$$

In spite of the importance and actuality there are only a few results about the initial-boundary value problem for pseudo-differential equations with nonanalytic symbols. For example, in paper [17] it was considered the case of rational symbol $K(p)$ with some poles in the right complex half-plane. A new method for constructing the Green operator was proposed. The core of this method is the introduction of certain necessary condition at the singularity points of the symbol $K(p)$. In paper [18] it was considered the initial-boundary value problem for a pseudo-differential equation with symbol $K(p) = |p|^{\frac{1}{2}}$. As far as we know the case of general nonhomogeneous nonanalytic symbols $K(p)$ was not studied previously. In the present paper we fill this gap, considering as example the Neumann problem (1.1), which plays an important role in contemporary mathematical physics. Note that the operator $-|\alpha|u_{xxx} + \int_0^{+\infty}\frac{\mathrm{sgn}(x-y)u_y(y,t)}{\sqrt{|x-y|}}\mathrm{d}y$ in Eq. (1.1) has a nonanalytic nonhomogeneous symbol $K(p) = -|\alpha|p^3 + \frac{p}{\sqrt{|p|}}$. So the general theory from book [16] is not applicable to our problem (1.1). There are many natural open questions which should be studied. First we need to answer the following question: how many boundary data should be posed in problem (1.1) for its correct solvability? We also study traditionally important problems in the theory of partial differential equations, such as existence and uniqueness of solutions. In order to construct Green operator we propose a new method based on the theory of singular integro-differential equations with Hilbert kernel. It will be shown that exactly two boundary values are necessary and sufficient to make the problem (1.1) well posed.

We believe that the results of this paper could be applicable to study a wide class of dissipative nonlinear equations with a fractional derivative on a half-line by the use of techniques of nonlinear analysis (estimations of function Green, fixed point theorems, etc., see [13]).

To state precisely the results of the present paper we introduce some notations. Define the Sobolev spaces

$$\mathbf{H}_p^s = \left\{\phi \in \mathbf{L}^p;\ \|f\|_{\mathbf{H}_p^s} = \|\partial_x^s f\|_{\mathbf{L}^p} < \infty\right\},$$

and $\mathbf{L}^p$ denotes the Lebesgue spaces, where

$$\|\phi\|_{\mathbf{L}^p} = \left(\int_0^\infty |f(x)|^p \mathrm{d}x\right)^{1/p}$$

and

$$\|\phi\|_{\mathbf{L}^\infty} = \operatorname{ess.}\sup_{x\in\mathbf{R}^+} |\varphi(x)|.$$