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The monotone iterative method and zeros of Bessel functions for nonlinear singular derivative dependent BVP in the presence of upper and lower solutions

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ABSTRACT

In this paper we consider a class of nonlinear singular boundary value problems

$$-\left(x^\alpha y'(x)\right)' + x^\alpha f\left(x, y(x), x^\alpha y'(x)\right) = 0, \quad 0 < x < 1,\ y'(0) = y'(1) = 0,$$

for $\alpha \geq 1$. We assume that the source function $f(x, y, x^\alpha y')$ is Lipschitz in $x^\alpha y'$ and one-sided Lipschitz in y. The initial approximations are an upper solution $u_0(x)$ and a lower solution $v_0(x)$ which can be ordered in one way, $v_0(x) \leq u_0(x)$, or the other, $u_0(x) \leq v_0(x)$. We propose an iterative scheme and establish the existence of solutions bounded by v_0 and u_0, and allow $\partial f/\partial y$ to take both positive and negative values. The method is constructive in nature and can be used to generate solutions of the nonlinear singular boundary value problems.

1. Introduction

The upper and lower solution technique is the most promising technique as far as singular boundary value problems are concerned [1]. Initially the shooting method [2] and nonlinear alternative methods [3] were used for treating singular nonlinear problems, though originally both methods were developed for treating non-singular problems. Nonlinear alternative methods can be further divided into topological degree theory ones [4] and topological transversality ones [5].

Nowadays, different techniques are being coupled with the upper and lower solution technique, e.g., those of the topological degree theory [6], topological transversality [7,8], the monotone iterative method [9–14] and quasilinearization [15].

In the present work we have utilized the upper and lower solution technique related to the monotone iterative method where the upper and lower solutions are well-ordered and where they are not well-ordered. For the non-singular case, i.e., $-y'' + f(x, y, y') = 0$, Cherpion et al. [9] considered the following iterative scheme:

$$-y''_{n+1} + \lambda y_{n+1} = -f(x, y_n, y'_n) + \lambda y_n, \qquad y'_{n+1}(0) = y'_{n+1}(1) = 0, \tag{1.1}$$

and proved that it is possible to make a good choice of λ such that the approximations converge monotonically to solutions of (1.1) for both well-ordered and non-well-ordered upper and lower solutions. They utilized the properties of $\cos\sqrt{|\lambda|}x(\sin\sqrt{|\lambda|}x)$ and $\cosh\sqrt{\lambda}x(\sinh\sqrt{|\lambda|}x)$ to make a good choice of λ.

Chawla and Shivkumar [10] considered the following class of nonlinear singular boundary value problems:

$$-(x^\alpha y'(x))' = x^\alpha f(x, y), \qquad 0 < x < 1,\ y'(0) = 0,\ y'(1) = 1 \tag{1.2}$$

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and proved the existence of the solutions for $u^* < \lambda_0$, where $u^* = \sup \partial f/\partial y$ and λ_0 is the first positive zero of $J_{(\alpha-1)/2}(\sqrt{\lambda})$ ($J_\nu(z)$ is the Bessel function of the first kind of order ν).

In this paper we consider the following class of nonlinear singular differential equations:

$$-\left(x^\alpha y'(x)\right)' + x^\alpha f\left(x, y(x), x^\alpha y'(x)\right) = 0, \quad 0 < x < 1, \ (\alpha \ge 1) \tag{1.3}$$

where the source function $f(x, y, x^\alpha y')$ is derivative dependent and the boundary conditions are of Neumann type, written as

$$y'(0) = 0, \qquad y'(1) = 0. \tag{1.4}$$

The nonlinear singular differential equations defined by Eq. (1.2) occur in several branches of the sciences, medical sciences and engineering [16–20]. There are several iterative schemes possible for the derivative dependent source function (see [9]). In the present work we look for an iterative scheme which is as simple as possible from the computational point of view. So we propose the following iterative scheme:

$$-\left(x^\alpha y_n'(x)\right)' + \lambda x^\alpha y_n(x) = -x^\alpha f\left(x, y_{n-1}, x^\alpha y_{n-1}'\right) + \lambda x^\alpha y_{n-1}(x) \tag{1.5}$$

$$y_n'(0) = 0, \qquad y_n'(1) = 0. \tag{1.6}$$

When $\alpha = 0$, i.e., for the non-singular case, we get the solutions of the corresponding linear differential equation as $\cos\sqrt{|\lambda|}x(\sin\sqrt{|\lambda|}x)$ for $\lambda < 0$ and $\cosh\sqrt{\lambda}x(\sinh\sqrt{|\lambda|}x)$ for $\lambda > 0$. For $\alpha \ge 1$ the corresponding singular linear differential equation has solutions in terms of Bessel functions. In the present work we have used the properties of Bessel functions. We prove some inequalities which involve Bessel functions to make a good choice of λ such that the approximations converge monotonically to the solutions of (1.3)–(1.4).

To start the iterations we assume the existence of the upper solution $u_0(x)$ and the lower solution $v_0(x)(\le u_0(x))$ which satisfy certain inequalities. The corresponding nonlinear operator is not necessarily monotone, but we prove that for $\lambda > 0$ large enough, sequences $\{u_n\}$ and $\{v_n\}$ generated by u_0 and v_0, respectively, converge monotonically to the solutions u and v of (1.3)–(1.4) such that $v_0 \le v \le u \le u_0$.

We also consider the case where the lower and upper solutions are in the reverse order, i.e., $u_0(x) \le v_0(x)$, and prove that the sequences generated from these converge to the solutions u and v of (1.3)–(1.4) such that $u_0 \le u \le v \le v_0$.

The work in this paper generalizes the work of Cherpion et al. [9] (for the non-singular case, $\alpha = 0$) to the singular case ($\alpha \ge 1$) and also generalizes the work of Chawla and Shivkumar [10] to derivative dependent source functions. Also the work in this paper does not require any of the sign restrictions on the nonlinear term $f(x, y, x^\alpha y')$ imposed by Dunninger and Kurtz [21], Bobisud [22] and Zhang [23]. This paper also improves the result of Ford and Pennline [24], as we allow $\partial f/\partial y$ to take both positive and negative values.

In Section 2 we discuss some elementary results, e.g., Lomel's transformation, maximum principles and the existence of two differential inequalities. Then using these elementary results we establish existence results for well-ordered upper and lower solutions in Section 3 and for reverse ordered upper and lower solutions in Section 4. In Section 5 we conclude this paper with some remarks.

2. Preliminaries

Suppose that $h(x) \in C[0, 1]$ and $\lambda \in \mathbb{R}_0$ $(\mathbb{R}_0 = \mathbb{R} \setminus \{0\})$, $A \in \mathbb{R}$ and $B \in \mathbb{R}$. Now, consider the following class of linear singular problems:

$$-\left(x^\alpha y'(x)\right)' + \lambda x^\alpha y(x) = x^\alpha h(x), \quad 0 < x < 1, \tag{2.1}$$

$$y'(0) = A, \qquad y'(1) = B. \tag{2.2}$$

The corresponding homogeneous system is given by

$$-\left(x^\alpha y'(x)\right)' + \lambda x^\alpha y(x) = 0, \quad 0 < x < 1, \tag{2.3}$$

$$y'(0) = 0, \qquad y'(1) = 0. \tag{2.4}$$

Using Lommel's transformation [10]

$$z = \beta\xi^\gamma, \qquad w = \xi^a v(\xi), \tag{2.5}$$

the standard Bessel equation

$$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} + (z^2 - \nu^2)w = 0, \tag{2.6}$$

is transformed into

$$\xi^2\frac{d^2v}{d\xi^2} + \xi(1-2a)\frac{dv}{d\xi} + \left[(\beta\gamma\xi^\gamma)^2 + (a^2 - \nu^2\gamma^2)\right]v = 0. \tag{2.7}$$