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# Global structure of Riemann solutions to a system of two-dimensional hyperbolic conservation laws[☆]

Chun Shen [a,*], Meina Sun [a,b], Zhen Wang [b]

[a] School of Mathematics and Information, Ludong University, Yantai 264025, PR China

[b] Wuhan Institute of Physics and Mathematics, The Chinese Academy of Sciences, Wuhan 430071, PR China

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ABSTRACT

The Riemann problem for a two-dimensional nonstrictly hyperbolic system of conservation laws is considered. Without the restriction that each jump of the initial data projects one planar elementary wave, ten topologically distinct solutions are obtained by applying the method of generalized characteristic analysis. Some of these solutions involve the nonclassical waves, i.e., the delta shock wave and the delta contact discontinuity, for which we explicitly give the expressions of their strengths, locations and propagation speeds. Moreover, we demonstrate that the nature of our solutions is identical with that of solutions to the corresponding one-dimensional Cauchy problem, which provides a verification that our construction produces the correct unique global solutions.



## 1. Introduction

The Riemann problem, a kind of Cauchy problem with the simplest discontinuous initial data, is the most fundamental problem in the field of nonlinear hyperbolic conservation laws. Compared to the Cauchy problem, it is much easier to study, but still reveals the basic properties of the Cauchy problem. Due to the explicit structure of the Riemann solutions, it also serves as a touchstone for numerical schemes.

For multi-dimensional systems of conservation laws, the well-known Euler system for compressible gases is the primary one. Many efforts have been devoted to the system, especially a body of work on the two-dimensional Riemann problem has been developed [1,2]. In [3], Zhang and Zheng investigated the four quadrant Riemann problem for the Euler equations, namely the initial data are four constant states in each quadrant of $(x, y)$ plane. With the method of generalized characteristic analysis, they constructed the boundaries of interaction of four planar waves from infinity case by case. In the domains of interaction, they formulated a set of conjectures of the wave patterns. Although some efforts have been made to prove these conjectures in the past twenty years, it is unfortunate that none of them has been proved rigorously. The Euler system remains formidable for its complexity. This motivates our interest in considering simplified models which can capture various isolated features of the Euler system. In this paper, we focus our attention on the following system of conservation

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* Corresponding author. Tel.: +86 535 6697510; fax: +86 535 6681264.
*E-mail addresses:* shenchun3641@sina.com (C. Shen), smnwhy0350@163.com (M. Sun), zhenwang@wipm.ac.cn (Z. Wang).

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law

$$\begin{cases} u_t + (u^2)_x + (u^2)_y = 0, \\ \rho_t + (\rho u)_x + (\rho u)_y = 0. \end{cases} \tag{1}$$

Eqs. (1) can be derived directly from the two-dimensional isentropic Euler equations

$$\begin{cases} \rho_t + (\rho u)_x + (\rho v)_y = 0, \\ (\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y = 0, \\ (\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y = 0, \end{cases} \tag{2}$$

by assuming that the speeds of the $x$-axis and $y$-axis direction are equal and furthermore both the pressure $p$ and the density $\rho$ are constants in the last two momentum equations.

On the other hand, it also comes from the system

$$\begin{cases} u_t + (uf(u, v))_x + (ug(u, v))_y = 0, \\ v_t + (vf(u, v))_x + (vg(u, v))_y = 0, \end{cases} \tag{3}$$

by letting $f(u, v) = g(u, v) = u$ and $v = \rho$. Equations like (3) occur in a variety of applications, including oil recovery, elastic theory and magneto-hydrodynamics [4]. The Riemann problem of (3) is much more complicated than the scalar case, but it is simpler than the problem for general hyperbolic system. This is due to the fact that the domain of mixed type does not appear in the study of self-similar solutions for (3). Thus the study of the Riemann problem for (3) can be regarded as a necessary step to more complicated and practical cases such as the conjectures on the two-dimensional Riemann solutions for the Euler equations [3]. This is why we are interested in studying the Riemann problem of type (3) and (1) is the simplest one of type (3).

When $f(u, v) = u$ and $g(u, v) = v$, the system (3) becomes

$$\begin{cases} u_t + (u^2)_x + (uv)_y = 0, \\ v_t + (uv)_x + (v^2)_y = 0, \end{cases} \tag{4}$$

which has been extensively studied. Tan and Zhang [5] firstly studied the Riemann problem of (4) and they discovered that the form of the standard Dirac delta function supported on a shock should be used as a part in their Riemann solutions for (4). We can see [6,7] for the related results about (4). At the mention of delta shock waves, we can also see [8–16] and the references cited therein.

It is worthwhile to notice that the system (1) also belongs to the particular triangular type [17] due to its special structure where the evolution of $u$ does not depend on the other unknown $\rho$. Moreover (1) may be viewed as an alternative way of writing a scalar conservation law with a discontinuous flux which has been widely studied recently. Obviously (1) is not strictly hyperbolic for its coinciding eigenvalues when $u = 0$. As a matter of fact, one cannot expect $\rho$ to be bounded variation and the measure should be considered. Thus we should deal with wave interactions including the delta shock wave for its resonant wave structure.

The aim of the present paper is to construct explicitly the global solutions to (1) with the Riemann initial data

$$(u, \rho)(t, x, y)\,|_{t=0} = (u_i, \rho_i), \quad \text{for } (x, y) \text{in the } i\text{th quadrant}, \tag{5}$$

where $(u_i, \rho_i),\ i = 1, 2, 3, 4$, are constant states. The four quadrant Riemann problems are sufficient to approximate general initial data using rectangular grids.

For most of the work on gas dynamics models, the Riemann initial data is restricted to satisfy the assumption that only a planar elementary wave appears at each interface of the initial data. In the present paper, we remove this restriction in the development of our solutions for the reason that generally an $n \times n$ system would be expected to develop $n$ waves from each initial discontinuity. Based on the method of generalized characteristic analysis, we solve the Riemann problem (1) and (5) analytically and ten exact entropy solutions with different geometric structures are constructed completely. The solutions reveal various interactions of waves involving not only the classical waves such as shock waves, rarefaction waves and contact discontinuities, but also the nonclassical waves as delta shock waves. A new kind of nonclassical wave, namely, the delta contact discontinuity appears in the interaction process, which is the Dirac delta function supported on the contact discontinuity. Another contribution of this paper is that the explicit expressions for the strengths of the delta shocks are provided in detail.

Moreover it can be verified that our solutions are globally unique by comparing with some one-dimensional rotated solutions. Although existence and uniqueness results for the solutions to systems in one-dimensional is known, there are no such results for systems in two-dimensional. Particularly no general uniqueness theory exists that is applicable to the system under consideration in this paper. However, under a rotation, the Riemann problem (1) and (5) can be converted into a one-dimensional Cauchy problem whose initial data can be described as the interacting Riemann problem [18–20]. Then a mapping between the one-dimensional Cauchy solutions and the two-dimensional Riemann solutions we constructed can be formally derived as in [19]. Examination of qualitative features of this mapping demonstrates the equality of these two solutions. This comparison provides a check that our construction produces the correct unique global solutions.

The rest of the paper is organized as follows. In Section 2, we provide some basic properties of system (1), including the characteristics, bounded discontinuities and planar elementary waves. Then we classify the Riemann problem according to the combinations of the planar elementary waves in Section 3. We spend the following two sections constructing the global