Nonlinear Analysis 74 (2011) 4338–4351

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Second-order optimality conditions using approximations for nonsmooth vector optimization problems under inclusion constraints[☆]

P.Q. Khanh [a], N.D. Tuan [b,*]

[a] Department of Mathematics, International University of Hochiminh City, Linh Trung, Thu Duc, Hochiminh City, Viet Nam

[b] Department of Mathematics and Statistics, University of Economics of Hochiminh City, 59C Nguyen Dinh Chieu, D.3, Hochiminh City, Viet Nam

ARTICLE INFO

Article history:
Received 26 February 2010
Accepted 18 March 2011
Communicated by Ravi Agarwal

MSC:
90C29
49J52
90C48
90C46

Keywords:
Nonsmooth vector optimization
Inclusion constraints
First and second-order approximations
Asymptotic p-compactness
Second-order optimality conditions

ABSTRACT

Second-order necessary conditions and sufficient conditions for optimality in nonsmooth vector optimization problems with inclusion constraints are established. We use approximations as generalized derivatives and avoid even continuity assumptions. Convexity conditions are not imposed explicitly. Not all approximations in use are required to be bounded. The results improve or include several recent existing ones. Examples are provided to show that our theorems are easily applied in situations where several known results do not work.

1. Introduction

Second-order optimality conditions are of great interest, since they refine first-order conditions with second-order information which is much helpful to recognize optimal solutions as well as to design numerical algorithms for computing them. In nonsmooth optimization, similarly as for first-order conditions, the major approach to second-order ones is using generalized derivatives to replace Fréchet and Gateaux derivatives which do not exist. However, at our disposal there are much fewer kinds of second-order derivatives than first-order ones. This paper is concerned with first and second-order optimality conditions using approximations introduced in [1,2] as generalized derivatives. In [3,2] first-order approximations were used together with regularity conditions to establish Karush–Kuhn–Tucker type first-order conditions. Functions and mappings involved in the considered problems were assumed to be locally Lipschitz or to have upper semicontinuous bounded-valued approximations. Second-order conditions using approximations were investigated in [1] assuming that all approximations in use were compact. Possibly unbounded approximations were employed to prove optimality conditions of both orders 1 and 2 in [4–8] for various vector optimization problems, including set-valued optimization. Semicontinuity requirements were not imposed either. Instead, asymptotic pointwise compactness of approximations was assumed. This compactness is relatively relaxed, since in the finite dimensional case any approximation is asymptotically compact (in finite dimensional spaces the word "pointwise" is omitted as pointwise convergence coincides

☆ This work was supported by the National Foundation for Science and Technology Development of Viet Nam.

* Corresponding author. Tel.: +84 0903382994.
E-mail address: ndtuan73@yahoo.com (N.D. Tuan).

0362-546X/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.na.2011.03.031

with norm convergence). Second-order optimality conditions for the optimistic case of bilevel optimization problems were developed in terms of approximations in [9], assuming also that the involved approximations were compact or bounded.

The aim of this note is to obtain second-order optimality conditions for the following vector problem under inclusion constraints. Throughout this paper, let X, Y and Z, if not otherwise stated, be normed spaces and $C \subseteq Y$ be a closed convex cone. Let $f : X \to Y$ be a (single-valued) mapping and $F : X \to 2^Z$ be a multifunction with closed convex values. Our problem is

$$\min f(x), \quad \text{s.t. } 0 \in F(x). \tag{P}$$

This problem seems to encompass most vector optimization problems, since the constraint is very general. If $F(x) = g(x)+K$, where g is single-valued and $K \subseteq Z$ is a convex cone, then the constraint is of the inequality form $g(x) \in -K$, which has been repeatedly considered in the literature. A regularity condition and Karush–Kuhn–Tucker first-order optimality conditions for problem (P) with $Y = \mathbb{R}^n$ and X, Z being Hilbert spaces were studied in [10], assuming that f and F are locally Lipschitz, and the Clarke subdifferential was applied. [3] extended this result, using semicontinuous bounded-valued first-order approximations. With $Y = \mathbb{R}$, $X = \mathbb{R}^n$ and $Z = \mathbb{R}^m$, [11] developed optimality conditions of order 2, using Clarke generalized Hessians, under the assumptions that f and the support function of F are $C^{1,1}$-functions. In [12] such optimality conditions for this scalar problem was investigated using compact approximations. Inspired by the above-mentioned papers we will investigate second-order optimality conditions for problem (P) under asymptotic pointwise compactness assumptions, trying to unify results in [4–6,12,13,11,14,15]. Comparisons, especially by examples, will show advantages of our results. Note that our optimality conditions are developed without even continuity and convexity assumptions.

The organization of the paper is as follows. In Section 2, we collect definitions and preliminary facts for our later use. Second-order optimality conditions for the case, where the involved maps have Fréchet derivative of order 1, are established in Section 3. Section 4 is devoted to the general nonsmooth case.

2. Preliminaries

Our notations are basically standard. $\mathbb{N} = \{1, 2, \dots, n, \dots\}$. For a normed space X, X^* stands for the topological dual of X; $\langle ., . \rangle$ is the canonical pairing. $\|.\|$ is used for the norm in any normed space (from the context no confusion occurs). $B_X(x, r) = \{z \in X \mid \|x - z\| < r\}$ and for $B_X(0, 1)$ we write simply B_X. $L(X, Y)$ denotes the space of the bounded linear mappings from X into Y and $B(X, X, Y)$ is that of the bounded bilinear mappings from $X \times X$ into Y. For a cone $C \subseteq X$, $C^* = \{c^* \in X^* \mid \langle c^*, c\rangle \geq 0, \forall c \in C\}$ is the polar cone of C and $C_1^* = \{c^* \in C^* | \ \|c^*\| = 1\}$. For $A \subseteq X$, int A, cl A, bd A and co A stand for the interior, closure, boundary and convex hull of A, respectively. For $t > 0$ and $k \in \mathbb{N}$, $o(t^k)$ designates a moving point such that $o(t^k)/t^k \to 0$ as $t \to 0^+$. $C^{1,1}$ is used for the space of the Fréchet differentiable mappings whose Fréchet derivative is locally Lipschitz.

Definition 2.1 (*[2,1]*). Let $x_0 \in X$ and $h : X \to Y$.

(i) A set $A_h(x_0) \subseteq L(X, Y)$ is called a first-order approximation of h at x_0 if there exists a neighborhood U of x_0 and positive r with $r\|x - x_0\|^{-1} \to 0$ as $x \to x_0$ such that, for all $x \in U$,

$$h(x) - h(x_0) \in A_h(x_0)(x - x_0) + rB_X.$$

(ii) A pair $(A_h(x_0), B_h(x_0))$, with $A_h(x_0) \subseteq L(X, Y)$ and $B_h(x_0) \subseteq B(X, X, Y)$, is said to be a second-order approximation of h at x_0 if $A_h(x_0)$ is a first-order approximation of h at x_0 and there is positive r with $r\|x - x_0\|^{-1} \to 0$ as $x \to x_0$ such that, for all $x \in U$,

$$h(x) - h(x_0) \in A_h(x_0)(x - x_0) + B_h(x_0)(x - x_0, x - x_0) + r^2B_X.$$

Remark 2.2. (i) If h has second Fréchet derivative $h''(x_0)$, then $(h'(x_0), \frac{1}{2}h''(x_0))$ is a second-order approximation of h.

(ii) (See [2,1]) If $h : \mathbb{R}^n \to \mathbb{R}^m$ is locally Lipschitz at x_0, then the Clarke Jacobian (see [16]) $\partial_C h(x_0)$ is a first-order approximation of h at x_0. If furthermore h is in $C^{1,1}$ at x_0, then $(h'(x_0), \frac{1}{2}\partial_C^2 g(x_0))$ is a second-order approximation of h at x_0, where $\partial_C^2 h(x_0)$ is the Clarke Hessian of h at x_0 (see [13]).

(iii) (See [4]) If $h : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and has a pseudo Jacobian mapping $\partial h(.)$ (see [17]) which is upper semicontinuous at x_0, then co$\partial h(x_0)$ is a first-order approximation of h at x_0. If h is continuously Fréchet differentiable in a neighborhood of x_0 and has a pseudo Hessian mapping $\partial^2 h(.)$ (see [18]) which is upper semicontinuous at x_0, then $(h'(x_0), \frac{1}{2}\,\text{co}\partial^2 h(x_0))$ is a second-order approximation of h at x_0.

So approximations are very general derivatives. Furthermore, each map has a trivial approximation at any point being the whole space. But even discontinuous maps may have nontrivial approximations. Simply think of function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^{-1}$ for nonzero x and $f(0) = 0$, which has an infinite discontinuity at zero but admits second-order approximation at zero $((\alpha, \infty), \{0\})$, for any positive α. However, we do not have uniqueness for approximations. In particular, any superset of an approximation is also an approximation.

Later, if P_n and P are in $L(X, Y)$ and P_n converges pointwise to P, then we write $P_n \xrightarrow{\text{p}} P$ or $P = \text{p-}\lim P_n$. A similar notation is adopted for $M_n, M \in B(X, X, Y)$.