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Existence and properties of solutions of a control system with hysteresis effect☆

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ABSTRACT

We consider a control system described by two ordinary nonlinear differential equations subject to a control constraint given by a multivalued mapping with closed nonconvex values, which depends on the phase variables. One of the equations contains the subdifferential of the indicator function of a closed convex set depending on the unknown phase variable. The equation containing the subdifferential describes an input–output relation of hysteresis type.

Along with the original control constraint, we also consider the convexified control constraint and the constraint consisting of the extremal points of the convexified control constraint.

We prove the existence of solutions of our control system with various control constraints and establish certain relationships between corresponding solution sets.

1. Introduction

Consider a nonlinear control system described by two ordinary differential equations of the following form

$$a_1(v(t), w(t))v'(t) + a_2(v(t), w(t))w'(t) = g(v(t), w(t))u^1(t) + c_1(v(t), w(t)), \tag{1.1}$$

$$b_1(v(t), w(t))v'(t) + b_2(v(t), w(t))w'(t) + \partial I_{v(t)}(w(t)) \ni h(v(t), w(t))u^2(t) + c_2(v(t), w(t)), \tag{1.2}$$

$$v(0) = v_0, \qquad w(0) = w_0, \quad t \in T = [0, 1],$$

subject to the control constraint

$$u(t) = (u^1(t), u^2(t)) \in U(t, v(t), w(t)) \quad \text{a.e. on } T. \tag{1.3}$$

Here $a_i(\cdot, \cdot)$, $b_i(\cdot, \cdot)$, $c_i(\cdot, \cdot)$, $i = 1, 2$, $g(\cdot, \cdot)$, $h(\cdot, \cdot)$ are scalar functions; for each $v \in \mathbb{R}$, $\partial I_v(\cdot)$ is the subdifferential of the indicator function $I_v(\cdot)$ of the interval $[f_*(v), f^*(v)] \subset \mathbb{R}$ with $f_*(\cdot)$ and $f^*(\cdot)$ being two nondecreasing functions such that $f_* \le f^*$ on $\mathbb{R}$; U is a multivalued mapping with closed bounded values; v_0 and w_0 are given numbers.

Along with (1.3) we will consider the following constraints

$$u(t) \in \operatorname{co} U(t, v(t), w(t)) \quad \text{a.e. on } T, \tag{1.4}$$

$$u(t) \in \operatorname{ext} \operatorname{co} U(t, v(t), w(t)) \quad \text{a.e. on } T, \tag{1.5}$$

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where $\mathrm{co}\, U(t, v, w)$ is the convex hull of the set $U(t, v, w)$ and ext $\mathrm{co}\, U(t, v, w)$ is the set of all extreme points of $\mathrm{co}\, U(t, v, w)$. Remark that in a finite-dimensional space the convex hull of a compact set is a convex compact set and

$$\mathrm{ext\,co}\ U(t, v, w) \subset U(t, v, w). \tag{1.6}$$

Note that the inclusion (1.2) describes plenty of physically relevant input–output relations $v \to w$ of hysteresis type. For instance, when $b_1 \equiv 0$, $b_2 \equiv 1$, $h \equiv 0$, $c_2 \equiv 0$ and $b_1 \equiv -1$, $b_2 \equiv 1$, $h \equiv 0$, $c_2 \equiv 0$, the inclusion (1.2) models the generalized play operator and the generalized stop operator, respectively (cf. [1–3]). These operators are typical examples of hysteresis input–output relations, and are used for the analysis of many nonlinear irreversible phenomena in nature such as solid–liquid phase transition with undercooling/superheating effect and martensite–austenite phase transition in shape memory alloys (cf. [4,5]).

In the present paper, we study the problem of existence of solutions of the system (1.1), (1.2) with different control constraints and we explore possible relations between corresponding solution sets. In particular, we establish the density of the set of solutions of the system (1.1), (1.2) with the control constraint (1.5) in the set of solutions of the same system with the control constraint (1.4). This property, called the ''bang-bang'' principle [6], plays a crucial role in developing numerical methods for solving optimal control problems based on Pontryagin's maximum principle.

Similar problems for control systems with subdifferential operators in a Hilbert space were considered in the articles [7–9] and others. However, in these works the subdifferential operators do not depend on the unknown variables. Note that when $u^1(\cdot) = u^2(\cdot) \equiv 1$ and $c_i(\cdot, \cdot) \equiv 0$, $i = 1, 2$, the problem of existence and uniqueness of solutions of Eqs. (1.1) and (1.2) was studied in [10]. In the existence part for the system (1.1), (1.2) with the control constraints (1.3)–(1.5) we follow this and a related article [11] in some technical aspects.

2. Preliminary notions and main assumptions

Let T be the interval $[0, 1]$ of the real line $\mathbb{R}$ with the Lebesgue measure μ and the σ-algebra Σ of μ-measurable subsets of T. Let $\|\cdot\|$ and $d(\cdot, \cdot)$ denote the norm and the distance, respectively, on the Euclidean space $\mathbb{R}^2$. The Hausdorff metric on the space of compact subsets from $\mathbb{R}^2$ we denote by $D(\cdot, \cdot)$. We follow [12] in defining various notions of measurability for a multivalued mapping. For a Banach space X the notation ω-X means that the space X is equipped with the weak topology. The same notation is used for subsets of the space X with the topology induced by that of the space ω-X.

On the space $L^2(T, \mathbb{R}^2)$ along with the standard norm we consider the following norm

$$\|f\|_\omega := \sup_{0 \le t \le t' \le 1} \left\| \int_t^{t'} f(s)\, \mathrm{d}s \right\|, \quad f \in L^2(T, \mathbb{R}^2). \tag{2.1}$$

We denote by $L^2_\omega(T, \mathbb{R}^2)$ the space $L^2(T, \mathbb{R}^2)$ with the norm (2.1). Let $m > 0$, then the Theorem of [13] says that the topologies of the spaces ω-$L^2(T, \mathbb{R}^2)$ and $L^2_\omega(T, \mathbb{R}^2)$ coincide on the set

$$S_m = \{f \in L^2(T, \mathbb{R}^2) : \|f(t)\| \le m \text{ for a.e. } t \in T\}. \tag{2.2}$$

Consequently, the set S_m is compact in the topology of the space $L^2_\omega(T, \mathbb{R}^2)$. The set S_m with the topology of the space $L^2_\omega(T, \mathbb{R}^2)$ we denote by S^ω_m.

We make the following assumptions on the functions appearing in the system (1.1)–(1.3).

Hypothesis $H(f)$**.** $f_*(v) \le f^*(v)$, $v \in \mathbb{R}$, and

(a) f_*, f^* are nondecreasing and Lipschitz continuous on $\mathbb{R}$;
(b) the derivatives $f'_* := \frac{\mathrm{d}f_*}{\mathrm{d}v}, f^{*\prime} := \frac{\mathrm{d}f^*}{\mathrm{d}v}$ are Lipschitz continuous on $\mathbb{R}$;
(c) $f_*(v) = f^*(v)$, $v \in (-\infty, -k_0] \cup [k_0, +\infty)$, for a positive number k_0.

Let

$$\mathcal{F} := \{(v, w) \in \mathbb{R}^2 : f_*(v) \le w \le f^*(v)\}. \tag{2.3}$$

Hypothesis $H(a_i, b_i, c_i, g, h)$**.** The functions $a_i, b_i, c_i : \mathcal{F} \to \mathbb{R}$, $i = 1, 2$, $g, h : \mathcal{F} \to \mathbb{R}$ have the following properties

(a) a_i, b_i, c_i, $i = 1, 2$, are Lipschitz continuous on $\mathcal{F}$,

$$a_1 \ge c_0, \qquad b_2 \ge c_0, \qquad a_2 \ge c_0 \quad \text{on } \mathcal{F} \text{ for a positive constant } c_0; \tag{2.4}$$

(b) $$a_1 b_2 - a_2 b_1 \ge c_0 \quad \text{on } \mathcal{F}; \tag{2.5}$$

(c) g, h are Lipschitz continuous on $\mathcal{F}$.

Hypothesis $H(U)$**.** U is a multivalued mapping from $T \times \mathcal{F}$ to $\mathbb{R}^2$ with compact values such that

(a) the mapping $t \mapsto \mathrm{co}\, U(t, v, w)$ is measurable;
(b) the mapping $(v, w) \mapsto \mathrm{co}\, U(t, v, w)$ is continuous in the Hausdorff metric $D(\cdot, \cdot)$ for a.e. $t \in T$;