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Invariant densities and escape rates: Rigorous and computable approximations in the L^∞-norm

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ABSTRACT

In this article, we study piecewise linear discretization schemes for transfer operators (Perron–Frobenius operators) associated with interval maps. We show how these can be used to provide rigorous **pointwise** approximations for invariant densities of Markov interval maps. We also derive the order of convergence of the approximate invariant density to the real one in the L^∞-norm. The outcome of this paper complements recent results on the formulae of escape rates of open dynamical systems, (Keller and Liverani, 2009) [7]. In particular, the novelty of our work over previous results on BV and L^∞ approximations is that it provides a method for explicit computation of the approximation error.

1. Introduction

Although this article is about the approximation of invariant densities for interval maps, it is intimately related to what are commonly termed *open dynamical systems* or maps with 'holes' [1,2]. Open dynamical systems have become a very active area of research. In part, this is due to their connection to metastable dynamical systems [3,4] and their applications in earth and ocean sciences [5,6]. Corresponding to invariant measures for closed dynamics, in open dynamical systems, long-term statistics are described by a *conditionally invariant measure* and its related *escape rate*, measuring the mass lost from the system per unit time [1].

In their recent article [7], Keller and Liverani obtained precise escape rate formulae for Lasota–Yorke maps with holes shrinking to a single point. These formulae depend, *pointwise*, on the invariant density of the corresponding closed system. Unfortunately, explicit formulae of invariant densities for Lasota–Yorke maps are, in general, unavailable. Thus, to complement the result of [7], it is natural to consider numerical schemes that provide rigorous and computable pointwise approximations of invariant densities.

In the literature, rigorous approximation results are available in the L^1-norm [8–11], the L^∞-norm [12] and in the BV-norm, the space of functions of bounded variation, [13,14]. None of these methods are well suited to our problem. For example, L^1 approximations cannot provide the pointwise information necessary for application of the formulae of [7] (see

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Section 7 in [15]). Previous convergence schemes in the L^∞-norm and BV-norm did not provide an explicit computation of the approximation error.[1]

By rigorous and computable approximation, we mean the following. Assume we are given a transformation τ (typically, a formula) and an error tolerance, for example $\Delta := 10^{-2}$. We choose a suitable discretization scheme for the transfer (Frobenius–Perron) operator associated to τ and we are asked to determine an explicit level of discretization, ϵ, such that the approximate invariant density f_ϵ^* for the discretized operator at level ϵ satisfies

$$\|f^* - f_\epsilon^*\|_\infty \leq \Delta. \tag{1.1}$$

Here, f^* is the invariant density for τ. We emphasize that, we assume the continuous density f^* to be unknown throughout this calculation. By computable we mean that, at each step, one can determine via an algorithm, within a finite number of steps, each quantity necessary to determine[2] f_ϵ^* and to guarantee inequality (1.1).

One novelty of our approach is that we will need to consider two different discretization schemes in order to carry out this task, both based on binned discretization of the state space. The results in this paper enable us to compute the number of bins m with $\epsilon = m^{-1}$, and the associated approximate density (as usual, denoted by f_m^*), which achieves the tolerance Δ, uniformly.

The first ingredient in our analysis uses a natural piecewise linear discretization scheme and the abstract perturbation result of [16]. The two Banach spaces involved in our computation are L^∞, and the space of Lipschitz continuous functions on the unit interval. The same Banach spaces were used in [17] to provide a computer-assisted estimate on the rate of decay of correlations. However, the discretization scheme which was used in [17] is the traditional Ulam method. Ulam's method does not fit in our setting, since it does not preserve the regularity of the space of Lipschitz continuous functions. The idea of our method is to first prove an appropriate Lasota–Yorke inequality for transfer operators associated with Markov interval maps, then, to construct a discretized transfer operator which preserves the regularity of the space of Lipschitz continuous functions and which is close, in some suitable norm, to the original transfer operator. Although, neither the original transfer operator nor its discretized counterpart is a contraction in the L^∞-norm[3], we obtain quasi-compactness of the original transfer operator, thanks to the result of [18]. We use the general setting of [16], which allows the study of perturbations of transfer operators, which are not necessarily contractions on either Banach space. With this machinery, we can compute an explicit upper bound on the norm of the resolvent, bounded away from the spectrum, of the transfer operator associated with the map. Once an upper bound on the norm of the resolvent is computed, we use a second discretization scheme, whose associated finite rank operator is Markov, to compute an approximate invariant density with the pre-specified error tolerance Δ.

The reason for using two different discretizations in our method is as follows. The first discretization has the projection property. This property is essential in the proofs related to the computation of an explicit upper bound on the norm of the resolvent. Moreover, it produces reasonable constants,[4] which are needed when using the perturbation result of [16]. However, this natural discretization leads to a non-Markovian finite rank operator. The lack of the Markov property makes the (theoretical) rate of convergence slow. Thus, at the next stage, we use a different discretization, which lacks the projection property,[5] but produces a finite rank operator which is Markov. With this Markov scheme we will obtain a computable rate of convergence which is of order $m^{-1} \ln m$.

In Section 2, we set up our notation and assumptions. We also recall known results on Markov interval maps which are needed in the sequel. In Section 3, we provide a Lasota–Yorke inequality on the space of Lipschitz continuous functions. In Section 4, we present our two discretization schemes and prove results about their regularity properties when acting on the space of Lipschitz continuous functions. In Section 5, we present the perturbation result of [16] as a sequence of steps which are necessary for rigorous computations and in Section 6, we apply the perturbation result to our problem. The main challenge of this paper lies in this section, where we design an algorithm which enables one to rigorously compute an upper bound on the norm of the resolvent of the continuous transfer operator. The resolvent estimate can then be used in Section 7 to compute the decay of correlations and to obtain a rate of convergence $Cm^{-1} \ln m$ in Section 8, where C is a computable constant. In Section 9, we implement the algorithm of Section 6 on a specific Markov interval map. Numerical results are reported for all critical constants. In particular, we compute the number of bins ($m = 7 \times 10^6$) that guarantee approximation of f^* by f_m^* within tolerance $\Delta = 10^{-2}$. Section 10 contains a discussion of an alternate projection-based (non-Markov) scheme and in general, on efficiency of both piecewise linear discretization schemes for uniform approximation.

1 See also footnote 20 in [10] about numerical obstacles when one attempts to obtain rigorous BV approximations by using traditional discretization schemes.

2 Of course, the efficiency of such an algorithm is an important issue. For our purpose, we will be satisfied with algorithms that can be implemented in standard mathematical software on a personal computer. Beyond that, we do not specifically address computational efficiency in this article.

3 Unlike the L^1 setting where the norm of the transfer operator is automatically $\|\mathcal{L}\|_1 \leq 1$, here we can only show that $|\mathcal{L}| \leq M$, where M is typically greater than or equal to 1.

4 This is very important from a computational point of view. In particular, smaller constants mean that less time will be spent on the computer to produce the desired computation.

5 Thus, we could not use the Markov discretization right from the beginning.