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New properties of the *D*-operator and its applications on the problem of periodic solutions to neutral functional differential system☆

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ABSTRACT

In this paper, the authors first investigate some new properties of the *D*-operator which is associated with neutral functional differential equations. Then, by using these new properties, the problem of existence of periodic solutions of the *D*-operator neutral functional differential system is studied. The interesting thing is that the *D*-operator is allowed to be unstable, which is generalized by the corresponding studies in the past under the crucial assumption that the *D*-operator is stable.

1. Introduction

In the theory of functional differential equations, the study of the existence of periodic solutions for neutral functional differential equations of *D*-operator type (NFDE(D)) in the following form:

$$\frac{\mathrm{d}Dx_t}{\mathrm{d}t} = f(t, x_t), \tag{1.1}$$

is a difficult topic. In this equation, $x_t(\theta) = x(t+\theta),\ \theta \in [-\tau, 0],\ \tau > 0$ is a constant, $D : C([-\tau, 0], R^n) \to R^n$ is linear, continuous, and atomic at zero, and $f \in C((R \times C[-\tau, 0], R^n), R^n)$ with $f(t+\omega, \varphi) \equiv f(t, \varphi)$ for all $\varphi \in C([-\tau, 0], R^n)$ and for any bounded set $\Omega \subset C([-\tau, 0], R^n)$, $f([0, \omega] \times \Omega)$ is bounded in R^n. The difficulty is mainly due to the fact that the message of how properties reflect some general properties of the solution is far from clear. For example, in the definition of a solution $u(t)$ of Eq. (1.1), it is only required that $D(u_t)$ is continuously differentiable in t, but, generally, $u(t)$ may not be differentiable in t [1–4]. In the foundation of periodic solution theory for NFDE(D), Jack Hale gave an important definition named stable *D*-operator [1,2]. The *D*-operator associated with Eq. (1.1) is called stable, if the zero solution of the functional equation $Dy_t = 0$, $y_0 = \varphi \in \{C([-\tau, 0], R^n) : D\varphi = 0\}$ is uniformly asymptotically stable.

Under the condition that the operator *D* is stable, Jack Hale obtained the following result [1].

Theorem A. *If $\Omega \subseteq C([-\tau, 0], R^n)$ is open, D is stable, then any ω-periodic solution to Eq. (1.1) has a continuous first derivative.*

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In the past years, many researchers [5–8] studied the problem of existence of periodic solutions to Eq. (1.1) by means of Theorem A and some fixed point theorems. Now, the question is that: If the D-operator is unstable, does any periodic solution to Eq. (1.1) has a continuous first derivative?

In the present paper, we study D-operator in the following form:

$$D : C([-\tau, 0], R^n) \to R^n, \qquad D(\varphi) = \varphi(0) - B\varphi(-\tau), \tag{1.2}$$

where $B = [b_{ij}]_{n\times n}$ is a real matrix, $\tau > 0$ is a constant. By using the functional analysis theory, some new properties of D-operator is investigated. For example, under the condition that the D-operator is allowed to be unstable, we obtain that any ω-periodic solution $u(t)$ to Eq. (1.1) is continuous differentiable in t. Then, as an application, we study the existence of periodic solution to Eq. (1.1) with D-operator defined by (1.2). The interesting thing is that, the D-operator is not required to be stable. So the work of the present paper is extended to the corresponding ones of [5–8]. Moreover, the matrix B may not be symmetrical, so that the results (Theorems 2.1 and 2.2) of the present paper are generalized to the corresponding ones of [9–14].

2. New properties of the *D*-operator

Let $a = (a_1, a_2, \ldots, a_n)^\top \in C^n$ be a complex vector, $|a| = \left(\sum_{i=1}^n |a_i|^2\right)^{1/2}$, and $H = [h_{ij}]_{n\times n}$, $|H| = \left(\sum_{i=1}^n \sum_{j=1}^n |h_{ij}|^2\right)^{1/2}$ be a complex matrix. Let

$$C_\omega = \{x : x \in C(R, R^n), x(t+\omega) \equiv x(t), \text{ for all } t \in R\},$$

with the norm $|\varphi|_{C_\omega} = \max_{t\in[0,\omega]} |\varphi(t)|$, and

$$C^1_\omega = \{x : x \in C^1(R, R^n), x(t+\omega) \equiv x(t), \text{ for all } t \in R\}$$

with the norm $|\varphi|_{C^1_\omega} = \max\{|\varphi|_{C_\omega}, |\varphi'|_{C_\omega}\}$.

$$P_\omega = \{x : x \in C(R, C), x(t+\omega) \equiv x(t)\}$$

with the norm $|\varphi|_\infty = \max_{t\in[0,\omega]} |\varphi(t)|$, and

$$P^1_\omega = \{x : x \in C^1(R, C), x(t+\omega) \equiv x(t)\}$$

with the norm $\|\varphi\| = \max\{|\varphi|_\infty, |\varphi'|_\infty\}$.

$$T_\omega := \{x = (x_1, x_2, \ldots, x_n) : x_i \in P_\omega,\ i = 1, 2, \ldots, n\}$$

with the norm defined by $|x|_{T_\omega} = \max_{t\in[0,\omega]} |x(t)|$; and

$$T^1_\omega := \{x = (x_1, x_2, \ldots, x_n) : x_i \in P^1_\omega,\ i = 1, 2, \ldots, n\}$$

with the norm defined by $|x|_{T^1_\omega} = \max\{|x|_{T_\omega}, |x'|_{T_\omega}\}$. Clearly, C_ω, C^1_ω, P_ω, P^1_ω, T_ω and T^1_ω are all Banach spaces.

From the definition of D-operator defined in (1.2), it is easy to see

$$D(x_t) = x(t) - Bx(t-\tau). \tag{2.1}$$

If set

$$A : C_\omega \to C_\omega, \qquad [Ax](t) = x(t) - Bx(t-\tau), \tag{2.2}$$

then for any $h \in C_\omega$, the existence of continuous ω-periodic solution $u(t)$ for the difference system

$$D(x_t) = h(t) \tag{2.3}$$

is equivalent to the existence of continuous ω-periodic solution $u(t)$ for the difference system

$$[Ax](t) = h(t). \tag{2.4}$$

Thus, in order to investigate the existence of continuous ω-periodic solutions to Eq. (2.3), it suffices for us to show that Eq. (2.4) has continuous ω-periodic solutions. Moreover, we further study some properties of the continuous ω-solution for Eq. (2.4).

Since $B = [b_{ij}]_{n\times n}$ is a real matrix, there must be a complex matrix U such that

$$UBU^{-1} = E_\lambda = \operatorname{diag}(J_1, J_2, \ldots, J_l) \tag{2.5}$$

is Jordan's normal matrix, where

$$J_i = \begin{pmatrix} \lambda_i & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & \lambda_i & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_i \end{pmatrix}_{n_i\times n_i}$$

with $\sum_{i=1}^l n_i = n$, $\{\lambda_i :\ i = 1, 2, \ldots, l\}$ is the set of eigenvalues of matrix B. Now, we set

$$A_i : P_\omega \to P_\omega, [A_i y](t) = y(t) - \lambda_i y(t-\tau), \quad \text{for all } t \in [0,\omega],\ i = 1, 2, \ldots, l. \tag{2.6}$$