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A coercive James's weak compactness theorem and nonlinear variational problems

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ABSTRACT

We discuss a perturbed version of James's sup theorem for weak compactness that not only properly generalizes that classical statement, but also some recent extensions of this central result: the sublevel sets of an extended real valued and coercive function whose subdifferential is surjective are relatively weakly compact. Furthermore, we apply it to generalize and unify some facts in mathematical finance and to prove that the unique possible framework in the development of an existence theory for a wide class of nonlinear variational problems is the reflexive one.

1. Introduction

The two most important results about weak compactness in a Banach space are Eberlein–Smuliam's and James's sup theorems. This latter asserts that a weakly closed subset A of a real Banach space E is weakly compact provided that each continuous and linear functional on E attains its supremum on A. In recent years, a few generalizations of James's sup theorem have appeared, some motivated by its use in mathematical finance, in which the linear optimization condition is replaced by another one of a perturbed nature; that is, for a fixed and adequate extended real-valued function f, $x^* - f$ attains its supremum, where x^* is any continuous and linear functional on E.

The first of these results deals with a specific subset of the space, its closed unit ball. Inspired by the fact that the set of norm attaining functionals in a real Banach space is not more than the range of the duality mapping, which in turn is the range of the subdifferential of a certain coercive, convex and lower semicontinuous function, Calvert and Fitzpatrick announced in [1,2] that a real Banach space is reflexive whenever its dual space coincides with the range of an extended real-valued coercive, convex and lower semicontinuous function whose effective domain has nonempty norm-interior. However, the erratum [1] makes [2] more difficult to follow, since the main addendum requires correcting non-written proofs of some statements in [2] which are adapted from [3].

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Subsequently, and for arbitrary subsets, in [4] it was proven that a closed and convex subset A of a real Banach space E is weakly compact each time there exists a bounded function $f : A \longrightarrow \mathbb{R}$ such that for all continuous and linear functional x^* on E, the function $x^*_{|A} - f$ attains its supremum on A. Finally, [5] contains another James's type result, but for a concrete class of Banach spaces and also under a certain boundedness assumption: if E is a separable real Banach space and $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a convex and lower semicontinuous function whose effective domain is bounded, and such that for all continuous and linear functional x^* on E, the function $x^* - f$ attains its supremum, then its sublevel sets are weakly compact. This same statement has been shown by Delbaen in [6] for a concrete nonseparable space of integrable functions.

In this paper we introduce a new version of James's theorem that generalizes all these results: if E is a real Banach space and $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a coercive function with subdifferential onto, then its sublevel sets are relatively weakly compact. As a consequence, we extend some well-known results in mathematical finance and prove that if for a real Banach space a suitable abstract optimization problem which includes lots of nonlinear or nonsmooth variational ones that arise in connection with numerous applied problems, in the theory of partial differential equations and many other areas of pure and applied mathematics admits a solution, then it is reflexive.

The organization of the paper is as follows. Section 2 is concerned with introducing some basic notions in extended real-valued functions, in particular that of coercivity, which appears with diverse meanings in the literature, establishing the mentioned version of James's theorem for coercive functions whose subdifferential is surjective, showing a topological property of the epigraph of such functions, and deducing the known results. Section 3 deals with applying our results to mathematical finance. Finally, in Section 4 we prove that reflexivity is the natural context where a variety of nonlinear variational results can be developed.

2. James's theorem for coercive functions

First of all, we give a brief review of elementary notions related to extended real-valued functions. Let E be a real Banach space. We denote its topological dual space by E^* and its closed unit ball by B_E. If $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a function, let us write $\operatorname{dom}(f)$ for its *effective domain*, that is,

$$\operatorname{dom}(f) := \{x \in E : f(x) < +\infty\}.$$

Given $x_0 \in E$, $\partial f(x_0)$ stands for the *subdifferential* of f at x_0, i.e., the subset of E^*

$$\{x^* \in E^* : \text{ for all } x \in E, x^*(x) - f(x) \leq x^*(x_0) - f(x_0)\}$$

if $x_0 \in \operatorname{dom}(f)$, while for $x_0 \notin \operatorname{dom}(f)$ it is simply $\emptyset$. Under additional assumptions of convexity and lower semicontinuity, some results such as the Brønsted–Rockafellar theorem [7, Theorem 3.17] or [8, Theorem 4.2.8] guarantee that for a certain $x \in \operatorname{dom}(f)$, $\partial f(x) \neq \emptyset$. The *range of the subdifferential* of f,

$$\{x^* \in E^* : \text{ there exists } x \in E \text{ with } x^* \in \partial f(x)\},$$

is denoted by $\partial f(E)$, and for a subset B of E^* we write

$$(\partial f)^{-1}(B) := \{x \in E : \text{ there exists } x^* \in B \text{ such that } x^* \in \partial f(x)\}.$$

Finally, the function f is said to be *proper* when its effective domain is nonempty, and *coercive* provided that

$$\lim_{\|x\| \to +\infty} \frac{f(x)}{\|x\|} = +\infty.$$

Obviously, such is the case when $\operatorname{dom}(f)$ is a bounded subset of E. It follows from the Brønsted–Rockafellar theorem that if f is coercive, then its subdifferential is large: $\partial f(E)$ is norm-dense in E^* (see [9, Theorem 2.3]). In our statements a stronger condition is assumed, that $\partial f(E) = E^*$, which is a perturbed optimization condition, since it is equivalent to the assertion

$$\text{for all } x^* \in E^*, \quad x^* - f \text{ attains its supremum on } E.$$

Theorem 1 focuses the main efforts that we must make in order to prove Theorem 2, which is our most important result. Before stating it, we present this easy result:

Lemma 1. *If E is a real Banach space, $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a function such that*

$$\textit{for all } x^* \in E^*, \quad x^* - f \textit{ is bounded above}$$

and A is a nonempty subset of E with $f(A)$ bounded above, then A is bounded.

Proof. To prove that A is bounded it suffices to check, in view of the uniform boundedness principle, that for all $x^* \in E^*$ the set $\{x^*(a) : a \in A\}$ is bounded above. Hence, let us consider a functional $x_0^* \in E^*$ for which, by hypothesis, there exists $\alpha \in \mathbb{R}$ such that

$$\text{for all } x \in E, \quad x_0^*(x) - f(x) \leq \alpha.$$

In particular,

$$\text{for all } a \in A, \quad x_0^*(a) \leq f(a) + \alpha,$$

and since $f(A)$ is bounded above, then the set $\{x_0^*(a) : a \in A\}$ is also bounded above. □