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Asymptotic behavior of positive solutions of a semilinear polyharmonic problem in the unit ball

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ABSTRACT

Let m be a positive integer. We investigate the existence and the asymptotic behavior of positive continuous solutions to the following semilinear polyharmonic boundary value problem in the unit ball B of $\mathbb{R}^n$ $(n \geq 2)$:

$$(-\Delta)^m u = a(x)u^\alpha, \qquad \lim_{|x|\to 1} \frac{u(x)}{(1-|x|)^{m-1}} = 0,$$

where $-1 < \alpha < 1$ and a is a positive measurable function in B such that there exists $c > 0$ satisfying for each $x \in B$,

$$\frac{1}{c} \leq a(x)(1-|x|)^\lambda \exp\left(-\int_{1-|x|}^{\eta} \frac{z(s)}{s} ds\right) \leq c,$$

$\eta > 1$, $\lambda \leq m(1+\alpha) + 1 - \alpha$ and z is a continuous function on $[0, \eta]$ with $z(0) = 0$.

1. Introduction

Let m be a positive integer and $B = \{x \in \mathbb{R}^n : |x| < 1\}$ be the unit ball of $\mathbb{R}^n$ $(n \geq 2)$. This paper deals with the existence and the asymptotic behavior of positive continuous solutions to the high order semilinear boundary value problem

$$\begin{cases} (-\Delta)^m u = au^\alpha & \text{in } B \text{ (in the sense of distributions)}, \\ \lim_{|x|\to 1} \dfrac{u(x)}{(1-|x|)^{m-1}} = 0, & \end{cases} \tag{1.1}$$

where $\alpha \in (-1, 1)$ and a is a positive measurable function on B satisfying an appropriate condition.

The pure elliptic equation

$$-\Delta u = a(x)u^\alpha, \quad \alpha < 1, \ x \in \Omega \subset \mathbb{R}^n, \tag{1.2}$$

has been extensively studied for both bounded and unbounded domains Ω in $\mathbb{R}^n$ $(n \geq 2)$. We refer the reader to [1–6], and the references therein, for various existence and uniqueness results related to solutions for the above equation with homogeneous Dirichlet boundary conditions.

Most recently, applying Karamata regular variation theory, many authors have studied the exact asymptotic behavior of solutions of Eq. (1.2); see for example [1,3,7–9]. For instance, in [8], Mâagli studied (1.2) in a bounded $C^{1,1}$-domain Ω, where

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the function a is in $C^{\gamma}_{loc}(\Omega)$ $(0 < \gamma < 1)$, such that there exists a constant $c > 0$ such that for each $x \in \Omega$,

$$\frac{1}{c} \leq a(x)\delta_{\Omega}^{\lambda}(x) \exp\left(-\int_{\delta_{\Omega}(x)}^{\eta} \frac{z(s)}{s} ds\right) \leq c,$$

where $\lambda \leq 2, \eta > \operatorname{diam}(\Omega), \delta_{\Omega}(x) = \operatorname{dist}(x, \partial\Omega)$ and z is a continuous function in $[0, \eta]$ with $z(0) = 0$ and $\int_0^{\eta} t^{1-\lambda} \exp\left(\int_t^{\eta} \frac{z(s)}{s} ds\right) dt < \infty$. Thanks to the sub–supersolutions method, Mâagli showed in [8] that (1.2) has a unique positive classical solution which satisfies homogeneous Dirichlet boundary conditions and gave sharp estimates on such a solution. This improved and extended the estimates stated in [2,7,5,6,9]. In this work, we extend the result of [8] to problem (1.1). As will be seen, this task presents some notable differences from the case $m = 1$. Indeed, to get an existence result for problem (1.2), we can use the sub–supersolutions method; see [10, Lemma 3]. Furthermore, the classical sub–supersolutions method is based on the fact that for a suitable nonnegative function c, the operator $\Delta u - c(x)u$ satisfies the maximum principle. This result is not available for higher order; see [11, Corollary 5.5]. So, it seems that we cannot apply the sub–supersolutions method to study problem (1.1), essentially for $m > 1$ and $\alpha < 0$. Therefore, we have to work around this obstacle and we shall use the Schauder fixed point theorem which requires invariance of a convex set under a suitable integral operator. Hence, we are restricted to considering only the case $-1 < \alpha < 1$. Note that the case $\alpha = 0$ plays a crucial role in the estimates of the solution of problem (1.1) (see Proposition 4).

Before presenting our main result, we would like to fix in the following some notation and make some assumptions. Throughout this paper, for $x \in \mathbb{R}^n$, $\delta(x) = 1 - |x|$ denotes the Euclidean distance between x and the boundary $\partial B = \{x \in \mathbb{R}^n : |x| = 1\}$. For two nonnegative functions f and g defined on a set S, the notation $f(x) \approx g(x)$, $x \in S$, means that there exists $c > 0$ such that $\frac{1}{c}f(x) \leq g(x) \leq cf(x)$, for all $x \in S$. Also, we use $\mathcal{K}$ to denote the set of all functions L defined on $(0, \eta]$, $\eta > 1$, by

$$L(t) := \exp\left(\int_t^{\eta} \frac{z(s)}{s} ds\right),$$

where $z \in C([0, \eta])$ such that $z(0) = 0$.

Remark 1. It is obvious that $L \in \mathcal{K}$ if and only if

$$L \text{ is a positive function in } C^1((0, \eta]) \text{ such that } \lim_{t \to 0^+} \frac{tL'(t)}{L(t)} = 0 \quad \text{and } L(\eta) = 1.$$

This implies, in particular, that if $L \in \mathcal{K}$, then for $\sigma > -1$, $\int_0^{\eta} t^{\sigma} L(t)\, dt < \infty$.

Our main purpose is to show the existence and the global behavior of a positive solution to problem (1.1). The function a is required to satisfy the following hypothesis:

(H) a is a positive measurable function on B satisfying for $x \in B$,

$$a(x) \approx (\delta(x))^{-\lambda} L(\delta(x)),$$

where $\lambda \leq m(1+\alpha) + 1 - \alpha$ and $L \in \mathcal{K}$ such that $\int_0^{\eta} t^{m(1+\alpha)-\alpha-\lambda} L(t)\, dt < \infty$.

Remark 2. According to Remark 1, we need to verify condition $\int_0^{\eta} t^{m(1+\alpha)-\alpha-\lambda} L(t)\, dt < \infty$ in hypothesis (H) only if $\lambda = m(1+\alpha) + 1 - \alpha$.

As a typical example of a function a satisfying (H), we cite:

Example 1. Let φ be the positive function defined on $(0, \eta]$ by

$$\varphi(t) = t^{-\lambda}\left(\log\left(\frac{2\eta}{t}\right)\right)^{-\mu}.$$

Then for $\lambda < m(1+\alpha) + 1 - \alpha$ and $\mu \in \mathbb{R}$ or $\lambda = m(1+\alpha) + 1 - \alpha$ and $\mu > 1$, the function

$$a(x) = \varphi(\delta(x)),$$

satisfies (H).

Now we are ready to present our main result.

Theorem 1. *Assume (H). Then problem (1.1) has a positive continuous solution u satisfying for $x \in B$,*

$$u(x) \approx \theta_{\lambda}(x), \tag{1.3}$$