Nonlinear Analysis 75 (2012) 651–661

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

On S-asymptotically ω-periodic functions and applications

Michelle Pierri

Departamento de Física e Matemática da Faculdade de Filosofia, Ciências e Letras de Ribeirão Preto, Universidade de São Paulo, CEP:14040-901, Ribeirão Preto, SP, Brazil

ARTICLE INFO

Article history:
Received 23 March 2010
Accepted 31 August 2011
Communicated by Enzo Mitidieri

Keywords:
S-asymptotically ω-periodic function
Abstract integral equations
Integro-differential equations

ABSTRACT

Let $(X, \|\cdot\|)$ be a Banach space and $\omega \in \mathbb{R}$. A bounded function $u \in C([0,\infty); X)$ is called S-asymptotically ω-periodic if $\lim_{t\to\infty}[u(t+\omega)-u(t)] = 0$. In this paper, we establish conditions under which an S-asymptotically ω-periodic function is asymptotically ω-periodic and we discuss the existence of S-asymptotically ω-periodic and asymptotically ω-periodic solutions for an abstract integral equation. Some applications to partial differential equations and partial integro-differential equations are considered.

1. Introduction

In this work, we continue our developments in [1–3] on S-asymptotically ω-periodic functions. Specifically, we establish conditions under which an S-asymptotically ω-periodic function is asymptotically ω-periodic and we study the existence of S-asymptotically ω-periodic and asymptotically ω-periodic solutions for a class of abstract integral equation of the form

$$u(t) = \mathcal{R}(t)x_0 + \int_0^t \mathcal{R}(t-s)G(s,u(s))ds, \quad t \geq 0, \tag{1.1}$$

where $(\mathcal{R}(t))_{t\geq 0}$ is a strongly continuous family of bounded linear operators on a Banach space $(X, \|\cdot\|)$, $x_0 \in X$ and $G : [0,\infty) \times X \to X$ is a continuous function.

The literature concerning S-asymptotically ω-periodic functions is very recent; see [1–10] among another works. As regards the qualitative properties of S-asymptotically ω-periodic functions, we cite the papers [1–3]. As regards the problem of the existence of S-asymptotically ω-periodic solutions for differential equations, we refer the reader to [4–8] for the case of ordinary differential equations described on finite dimensional spaces and to [1,3,9,10] for differential equations defined on abstract Banach spaces.

Next, we include some definitions, properties and technicalities needed to establish our results. Let $(Z, \|\cdot\|_Z)$ and $(W, \|\cdot\|_W)$ be Banach spaces. The notation $\mathcal{L}(Z, W)$ is used to represent the space of bounded linear operators from Z into W endowed with the uniform operator norm denoted by $\|\cdot\|_{\mathcal{L}(Z,W)}$, and we write simply $\mathcal{L}(Z)$ and $\|\cdot\|_{\mathcal{L}(Z)}$ when $Z = W$. In this paper, $C_b([0,\infty), Z)$, $C_0([0,\infty), Z)$ and $C_\omega([0,\infty), Z)$ are the spaces

$$C_b([0,\infty), Z) = \left\{ x \in C([0,\infty), Z) : \sup_{t\geq 0} \|x(t)\| < \infty \right\},$$

$$C_0([0,\infty), Z) = \left\{ x \in C_b([0,\infty), Z) : \lim_{t\to\infty} \|x(t)\| = 0 \right\},$$

$$C_\omega([0,\infty), Z) = \{ x \in C_b([0,\infty), Z) : x \text{ is } \omega\text{-periodic} \},$$

endowed with the norm of the uniform convergence denoted by $\|\cdot\|_{C_b(Z)}$.

E-mail address: michellepierri@yahoo.com.br.

0362-546X/$ – see front matter
doi:10.1016/j.na.2011.08.059

Definition 1.1. A function $f \in C_b(\mathbb{R}, Z)$ is called almost periodic if for every $\varepsilon > 0$, there exists a relatively dense subset $\mathcal{H}(\varepsilon, f)$ of $\mathbb{R}$ such that $\|f(t+\xi) - f(t)\| < \varepsilon$ for every $t \in \mathbb{R}$ and all $\xi \in \mathcal{H}(\varepsilon, f)$.

Definition 1.2. A function $f \in C_b([0,\infty), Z)$ is called asymptotically almost periodic if there exist an almost periodic function $z(\cdot)$ and $w \in C_0([0,\infty), Z)$ such that $f = z + w$. If $z(\cdot)$ is ω-periodic, $f(\cdot)$ is said to be asymptotically ω-periodic.

Definition 1.3. A function $f \in C_b([0,\infty), Z)$ is said to be S-asymptotically periodic if there exists $\omega \in \mathbb{R}$ such that $\lim_{t\to\infty}[f(t+\omega) - f(t)] = 0$. In this case, we say that ω is an asymptotic period of $f(\cdot)$ and that $f(\cdot)$ is S-asymptotically ω-periodic.

Throughout this paper, $SAP_\omega(Z)$ represents the space formed for all the Z-valued S-asymptotically ω-periodic functions provided with the uniform convergence norm.

In the remainder of this paper, $(X, \|\cdot\|)$ is a Banach space and ω is a fixed positive real number. In addition, for $t \geq 0$, we use the decomposition $t = \xi(t) + \tau(t)\omega$ where $\xi(t) \in [0,\omega)$ and $\tau(t) \in \mathbb{N} \cup \{0\}$. Moreover, for $h \geq 0$ and $f \in C_b([0,\infty), Z)$, we denote by f_h the function $f_h : [0,\infty) \to Z$ defined by $f_h(t) = f(t+h)$.

This paper has four sections. In the next section, we establish conditions under which an S-asymptotically ω-periodic function is asymptotically ω-periodic. In Section 3, we discuss the existence of S-asymptotically ω-periodic solutions and asymptotically ω-periodic solutions for Eq. (1.1). Finally, in Section 4, some concrete applications to partial differential equations and partial integro-differential equations are considered.

2. On S-asymptotically ω-periodic functions

The study of the relations between the classes of S-asymptotically ω-periodic functions and asymptotically ω-periodic functions was initiated in [6]. In [6] it is established that all S-asymptotically ω-periodic function is asymptotically ω-periodic. However, in [1,2] there are presented some examples of S-asymptotically ω-periodic functions which are not asymptotically ω-periodic. Motivated by the above, in this section we establish conditions under which an S-asymptotically ω-periodic function is asymptotically ω-periodic. To begin, we consider the following definition introduced in [1].

Definition 2.4. A function $f \in C_b([0,\infty), Z)$ is said to be ω-normal on compact sets if for every sequence of natural numbers $(m_n)_{n\in\mathbb{N}}$ with $m_n \to \infty$ as $n \to \infty$, there exist a subsequence $(m_{n_j})_{j\in\mathbb{N}}$ and $F \in C_b([0,\infty), Z)$ such that $f_{m_{n_j}\omega} \to F$ as $j \to \infty$ uniformly on compact subsets of $[0,\infty)$.

From [1] we also include the next result.

Lemma 2.1. *Let $f \in SAP_\omega(Z)$ and assume that there exist $F \in C_b([0,\infty), Z)$ and a sequence of positive numbers $(t_n)_{n\in\mathbb{N}}$ with $t_n \to \infty$ as $n \to \infty$, such that $f_{t_n} \to F$ uniformly on compact subsets of $[0,\infty)$. Then F is ω-periodic.*

Now, we can establish now our first result.

Proposition 2.1. *Let $f \in SAP_\omega(X)$ and $P : [0,\infty) \to \mathbb{R}$ be the function defined by $P(t) = \|f(t+\omega) - f(t)\|$. Assume that $f(\cdot)$ is ω-normal on compact sets, $P(\cdot)$ is non-increasing and $\sum_{j=1}^{\infty} P(j\omega) < \infty$. Then $f(\cdot)$ is asymptotically ω-periodic.*

Proof. By noting that $f \in SAP_\omega(X)$ and f is ω-normal on compact sets, from Lemma 2.1 we infer that there exist a subsequence $(n_j)_{j\in\mathbb{N}}$ of $(n)_{n\in\mathbb{N}}$ and $F \in C_\omega([0,\infty), X)$ such that $f_{n_j\omega} \to F$ uniformly on compact subsets of $[0,\infty)$. For $\varepsilon > 0$, we select $j_0 \in \mathbb{N}$ such that

$$\sum_{i \geq n_{j_0}} \|f(i\omega + \omega) - f(i\omega)\| \leq \varepsilon, \tag{2.2}$$

$$\|F(s) - f(s + n_j\omega)\| \leq \varepsilon, \quad \forall s \in [0,\omega],\ \forall j \geq j_0. \tag{2.3}$$

For $t > n_{j_0}\omega$, we have that $\eta(t) = n_{j_0} + h$ for some $h \in \mathbb{N}$. Now, from (2.2)–(2.3) we get

$$\begin{aligned}
\|F(t) - f(t)\| &= \|F(\xi(t) + \eta(t)\omega) - f(\xi(t) + \eta(t)\omega)\| \\
&\leq \|F(\xi(t)) - f(\xi(t) + n_{j_0}\omega)\| + \|f(\xi(t) + n_{j_0}\omega) - f(\xi(t) + (n_{j_0} + h)\omega)\| \\
&\leq \varepsilon + \sum_{i=n_{j_0}}^{n_{j_0}+h-1} \|f(\xi(t) + (i+1)\omega) - f(\xi(t) + i\omega)\| \\
&\leq \varepsilon + \sum_{i \geq n_{j_0}} \|f(i\omega + \omega) - f(i\omega)\| \\
&\leq 2\varepsilon,
\end{aligned}$$

which implies that $\lim_{s\to\infty}[F(s) - f(s)] = 0$ and $f(\cdot)$ is asymptotically ω-periodic since $f = F + (f - F)$. The proof is complete. □

Proposition 2.2. *Let $f \in SAP_\omega(X)$. Assume that $f(\cdot)$ is ω-normal on compact sets and $f \in W^{1,1}([0,\infty), X)$. Then $f(\cdot)$ is asymptotically ω-periodic.*