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Critical point theorems in cones and multiple positive solutions of elliptic problems

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ABSTRACT

We obtain critical point variants of the compression fixed point theorem in cones of Krasnoselskii. Critical points are localized in a set defined by means of two norms. In applications to semilinear elliptic boundary value problems this makes possible the use of local Moser–Harnack inequalities for the estimations from below. Multiple solutions are found for problems with oscillating nonlinearity.

1. Introduction

There exists a huge literature devoted to the existence and localization of positive solutions of various types of integral, and ordinary differential and partial differential equations. One of the most common approaches is based on Krasnoselskii's fixed point theorem in cones (see [1–4]) and has been intensively used in studying boundary value problems for ordinary differential equations and integral equations (see [5–8,2,9–11,4,12,13]). Its success is due to the upper and lower inequalities for the appropriate Green's functions. Similar inequalities for boundary value problems related to partial differential equations are not known and Krasnoselskii's theorem has appeared quite inapplicable to such problems. Some progress in this direction has been made in [14,15], where bilateral estimates are used only with respect to one of the variables (say, the time variable), or, by iteration, successively, to all of the variables. Obviously, this has required a suitable geometry of the domain of the equation. Also in paper [16] we pointed out the role of global weak Harnack inequalities for nonnegative superharmonic functions in the applicability of Krasnoselskii's theorem to semilinear elliptic problems in general domains. However, global Harnack inequalities in n dimensions, $n > 1$, are not known and therefore we may think of using instead local Harnack inequalities. The aim of this paper is to show that local Moser–Harnack inequalities can be used together with variational versions of Krasnoselskii's theorem in order to prove the existence, localization and multiplicity of positive solutions for semilinear elliptic problems. The results in this paper extend and complement those from [17–19,16,20–22] and make clear some arguments first used in [23].

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2. Preliminaries

Consider a real Hilbert space X with inner product and norm $(.,.)$, $|.|$, and a Banach space H with norm $\|.\|$, and assume that there exists a linear continuous map $\mathcal{I} : X \to H$ by which we shall identify X with the linear subspace $\mathcal{I}(X)$ of H, and any element $u \in X$ with its image $\mathcal{I}u \in H$. Thus we shall say that $X \subset H$ and the embedding is continuous. When $\mathcal{I}$ is compact we say that the embedding is compact. Let us denote by c_0 the best embedding constant with

$$\|u\| \le c_0|u| \quad \text{for all } u \in X, \tag{2.1}$$

that is $c_0 = \sup\{\|u\| : u \in X, |u| = 1\}$.

We consider a C^1 real functional E defined on X and we are interested in the equation

$$E'(u) = 0.$$

By a cone of X we shall understand a convex closed nonempty set $K \subset X$, $K \neq \{0\}$, with $\lambda u \in K$ for every $u \in K$ and $\lambda \ge 0$, and $K \cap (-K) = \{0\}$. Let $\phi \in K \setminus \{0\}$ be a fixed element with $|\phi| = 1$. Then, for all numbers R_0, R_1 with $0 < R_0 < \|\phi\| R_1$, there is $\mu > 0$ such that $\|\mu\phi\| > R_0$ and $|\mu\phi| < R_1$. Denote by $K_{R_0R_1}$ the connected component of the set $\{u \in K : \|u\| \ge R_0,\ |u| \le R_1\}$ which contains $\mu\phi$. Clearly $\mu\phi$ is an interior point of $K_{R_0R_1}$. We note that, in particular, when $X = H$ and $\|.\| = |.|$, $K_{R_0R_1}$ is the conical shell $\{u \in K : R_0 \le |u| \le R_1\}$.

By $\langle .,. \rangle$ we denote the natural duality between X and X', that is $\langle u^*, u\rangle = u^*(u)$ for $u \in X$ and $u^* \in X'$, and also the natural duality between H and H'. We consider a C^1 normalization function $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$, i.e. continuously differentiable, with $\varphi(0) = 0$, $\varphi'(\tau) > 0$ for every $\tau > 0$ and $\varphi(\tau) \to \infty$ as $\tau \to \infty$, and we denote by P the duality map on H associated with φ, assumed to be a continuous single-valued map, namely $P : H \to H'$,

$$\langle Pu, u\rangle = \|Pu\|_{H'}\|u\|, \qquad \|Pu\|_{H'} = \varphi(\|u\|) \quad \text{for all } u \in H.$$

We shall restrict ourselves to spaces H with the additional property that for each function $u \in C^1(\mathbf{R}_+, H)$, function $Pu(t)$ belongs to $C^1(\mathbf{R}_+, H')$ and

$$\left\langle \frac{d}{dt}(Pu(t)), u(t)\right\rangle = \varkappa\langle Pu(t), u'(t)\rangle \quad \text{for all } t \in \mathbf{R}_+$$

and some constant $\varkappa \in \mathbf{R}_+$. These guarantee that

$$\begin{aligned}[\varphi(\|u(t)\|) + \varphi'(\|u(t)\|)\|u(t)\|]\frac{d}{dt}\|u(t)\| &= \langle Pu(t), u'(t)\rangle + \left\langle \frac{d}{dt}(Pu(t)), u(t)\right\rangle \\ &= (1+\varkappa)\langle Pu(t), u'(t)\rangle, \end{aligned} \tag{2.2}$$

which shows that $\langle Pu(t), u'(t)\rangle$ and $\frac{d}{dt}\|u(t)\|$ have the same sign.

Let L be the linear operator from X to X' (the canonical isomorphism of X onto X'), given by

$$(u, v) = \langle Lu, v\rangle \quad \text{for all } u, v \in X,$$

and let J from X' into X be the inverse of L. Then

$$(Ju, v) = \langle u, v\rangle \quad \text{for all } u \in X', v \in X.$$

From

$$|Ju|^2 = (Ju, Ju) = \langle u, Ju\rangle \le |u|_{X'}|Ju|,$$

$(u \in X')$, we have that $|Ju| \le |u|_{X'}$, which shows that J is a linear continuous map. Also, if $u \in H$, then

$$|JPu|^2 = (JPu, JPu) = \langle Pu, JPu\rangle \le \|Pu\|_{H'}\|JPu\| = \varphi(\|u\|)\|JPu\| \le c_0\varphi(\|u\|)|JPu|.$$

Hence

$$|JPu| \le c_0\varphi(\|u\|). \tag{2.3}$$

In addition there exists a number R with $R \le R_1$ and

$$|JPu| \ge R > 0 \quad \text{for all } u \in K_{R_0R_1}. \tag{2.4}$$

Indeed, otherwise, there would be a sequence (u_k) of elements in $K_{R_0R_1}$ with $|JPu_k| \to 0$ as $k \to \infty$. Now, from

$$\varphi(R_0)R_0 \le \varphi(\|u_k\|)\|u_k\| = \langle Pu_k, u_k\rangle = (JPu_k, u_k) \le |JPu_k|\,|u_k| \le R_1|JPu_k|$$

letting $k \to \infty$, we derive the contradiction $\varphi(R_0)R_0 \le 0$.

We shall assume that JP is positive with respect to K, i.e.

$$JP(K) \subset K.$$