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On nonlinear distributional and impulsive Cauchy problems

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ABSTRACT

In this paper, existence results are derived for the unique, smallest, greatest, minimal and maximal solutions of nonlinear distributional Cauchy problems. Dependence of solutions on the data is also studied. The obtained results are applied to impulsive differential equations. Main tools are fixed point results in function spaces and recently introduced concepts of regulated and continuous primitive integrals of distributions.

1. Introduction

The main purpose of this paper is to prove new existence, uniqueness and extremality results for solutions of the nonlinear distributional Cauchy problem

$$y' = f(y), \qquad y(a) = c. \tag{1.1}$$

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Novel results for dependence of solutions on f and c are also derived. The values of f are distributions on a compact interval $[a, b]$ of $\mathbb{R}$. Solutions of (1.1) are assumed to be in the space $\mathcal{R}[a, b]$ of those functions from $[a, b]$ to $\mathbb{R}$ which are left-continuous on $(a, b]$, right-continuous at a, and which have right limits at every point of (a, b). With this presupposition, the Cauchy problem (1.1) is transformable into an integral equation that includes the regulated primitive integral of distributions introduced recently in [1].

The paper is organized as follows. Distributions on $[a, b]$, their primitives, the regulated primitive integral and some of their properties needed in this paper are considered in Section 2. Conversion of (1.1) into an integral equation and some fixed point results are also presented. Section 3 is devoted to the study of conditions under which (1.1) is well-posed, i.e., it has in $\mathcal{R}[a, b]$ a unique solution that depends continuously on the initial value c. Proofs are based on a generalized version of [2, Theorem 1.4.9]. It is proved in Section 9.

In Section 4, existence and comparison results are derived for the smallest and greatest solutions of the Cauchy problem (1.1). The case when solutions are also monotone is treated separately in Section 5. In Section 6, existence results are established for minimal and maximal solutions of the Cauchy problem (1.1). Section 7 begins with the proof of a new uniqueness lemma in $\mathcal{R}[a, b]$. An application to the Cauchy problem (1.1) is given.

A fact that makes the solution space $\mathcal{R}[a, b]$ important in applications is that it contains primitives of Dirac delta distributions $\delta_\lambda, \lambda \in (a, b)$. This fact is exploited in Section 8, where results derived for (1.1) are applied to present new results for impulsive differential equations. Also the continuous primitive integral of distributions introduced in [3] is used in these applications. This integral is used in [4] for the study of continuous solutions of distributional Cauchy problem (1.1).

Main tools are fixed point results proved in this paper and in [5,2] by ordinary and generalized iteration methods. Concrete examples are solved to illustrate the obtained results. Iteration methods, induction and/or Maple programming are used to determine solutions.

2. Preliminaries

Distributions on a compact real interval $[a, b]$ are (cf. [6]) continuous linear functionals on the topological vector space $\mathcal{D}$ of functions $\varphi : \mathbb{R} \to \mathbb{R}$ possessing for every $j \in \mathbb{N}_0$ a continuous derivative $\varphi^{(j)}$ of order j that vanishes on $\mathbb{R} \setminus (a, b)$. The space $\mathcal{D}$ is endowed with the topology in which the sequence (φ_k) of $\mathcal{D}$ converges to $\varphi \in \mathcal{D}$ if and only if $\varphi_k^{(j)} \to \varphi^{(j)}$ uniformly on (a, b) as $k \to \infty$ and $j \in \mathbb{N}_0$. As for the theory of distributions, see, e.g., [7,8].

In this paper, every distribution g on $[a, b]$ is assumed to have a primitive, i.e., a function $G : [a, b] \to \mathbb{R}$ whose distributional derivative G' equals to g, in the function space

$$\mathcal{R}[a, b] = \{G : \lim_{t\to s+} G(s) \text{ exists}, \lim_{s\to t-} G(s) = G(t) \text{ if } a \le s < t \le b, \text{ and } G(a) := \lim_{s\to a+} G(s)\}. \tag{2.1}$$

The value $\langle g, \varphi\rangle$ of g at $\varphi \in \mathcal{D}$ is thus given by

$$\langle g, \varphi\rangle = \langle G', \varphi\rangle = -\langle G, \varphi'\rangle = -\int_a^b G(t)\varphi'(t)\, dt.$$

Such a distribution g is called *RP* integrable. Its regulated primitive integral is defined by

$${}^r\!\!\int_s^t g := G(t) - G(s), \quad a \le s \le t \le b, \text{ where } G \text{ is a primitive of } g \text{ in } \mathcal{R}[a, b]. \tag{2.2}$$

Denote by $\mathcal{A}_R[a, b]$ the set of those distributions on $[a, b]$ that are *RP* integrable on $[a, b]$. If $g \in \mathcal{A}_R[a, b]$, then the function $t \mapsto {}^r\!\!\int_a^t g$ is that primitive of g which belongs to the set

$$\mathcal{P}_R[a, b] = \{G \in \mathcal{R}[a, b] : G(a) = 0\}.$$

It can be shown (cf. [1]) that a relation $\preceq$, defined by

$$f \preceq g \text{ in } \mathcal{A}_R[a, b] \quad \text{if and only if} \quad {}^r\!\!\int_a^t f \le {}^r\!\!\int_a^t g \quad \text{for all } t \in [a, b], \tag{2.3}$$

is a partial ordering on $\mathcal{A}_R[a, b]$. In particular,

$$f = g \text{ in } \mathcal{A}_R[a, b] \quad \text{if and only if} \quad {}^r\!\!\int_a^t f = {}^r\!\!\int_a^t g \quad \text{for all } t \in [a, b]. \tag{2.4}$$

The modulus $|g|$ of a distribution $g \in \mathcal{A}_R[a, b]$ is defined by

$$|g| := \sup\{g, -g\}, \tag{2.5}$$

where the supremum is taken in the partial ordering $\preceq$ defined by (2.3). $|g|$ exists because $\preceq$ is a lattice ordering on $\mathcal{A}_R[a, b]$ (cf. [1, Sect. 9]).