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Existence and uniqueness for nonlinear elliptic equations with lower-order terms

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ABSTRACT

We study the nonlinear Dirichlet problem for the elliptic equation

$$\operatorname{div}(\mathcal{A}(x, \nabla u) + b(x, u)) = \operatorname{div} F$$

in a regular domain $\Omega \subset \mathbb{R}^N$, $N > 2$, $u = 0$ on $\partial\Omega$. In hypothesis of Lipschitz continuity and strong monotonicity of $\mathcal{A}$, we assume that the lower order term $b(x, s)$ verify

$$|b(x, s) - b(x, t)| \leqslant E(x)|s - t|$$

for a.e. $x \in \Omega$ and for any $s, t \in \mathbb{R}$, where E is a non negative function in the Lorentz space $L^{N,q}(\Omega)$, $N \leqslant q \leqslant +\infty$. Without any control on the norm of E, with $F \in L^p$, $p > 1$ and $q < \infty$, we obtain existence and uniqueness result for distributional solutions $u \in W^{1,p}(\Omega)$ whenever p is close to two. For $q = \infty$, uniqueness results are obtained.

The main difficulty to solve the problem is due to noncoercivity of the vector field $\mathscr{A}(x, s, \xi) = \mathcal{A}(x, \xi) + b(x, s)$. Moreover, no classical structure conditions are satisfied.

1. Introduction

In this paper, we consider the Dirichlet problem

$$\begin{cases} \operatorname{div}(\mathcal{A}(x, \nabla u) + b(x, u)) = \operatorname{div} F & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with C^1-boundary, $N > 2$. Here $\mathcal{A} : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Caratheòdory function i.e.

$$x \to \mathcal{A}(x, \xi) \quad \text{is measurable for any } \xi \in \mathbb{R}^N; \tag{1.2}$$

$$\xi \to \mathcal{A}(x, \xi) \quad \text{is continuous for almost every } x \in \Omega. \tag{1.3}$$

We assume that there exist $0 < \alpha < \beta$ such that for almost every $x \in \Omega$ we have

$$|\mathcal{A}(x, \xi) - \mathcal{A}(x, \eta)| \leqslant \beta|\xi - \eta| \quad \text{(Lipschitz continuity)} \tag{1.4}$$

$$\alpha|\xi - \eta|^2 \leqslant \langle \mathcal{A}(x, \xi) - \mathcal{A}(x, \eta), \xi - \eta \rangle \quad \text{(strong monotonicity)} \tag{1.5}$$

$$\mathcal{A}(x, 0) = 0 \tag{1.6}$$

for any vectors ξ and η in $\mathbb{R}^N$. Moreover, we assume that $b : \Omega \times \mathbb{R} \to \mathbb{R}^N$ is a Caratheòdory function verifying the following properties:

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(i) There exists a non negative function $E : \Omega \to \mathbb{R}_+$ in the Lorentz space $E \in L^{N,q}(\Omega)$, $N \leqslant q \leqslant +\infty$, such that

$$|b(x, s) - b(x, t)| \leqslant E(x)|s - t|, \tag{1.7}$$

for a.e. $x \in \Omega$ and for any $s, t \in \mathbb{R}$.

(ii) For some $r > 2$,

$$b_0(x) := b(x, 0) \in L^r(\Omega, \mathbb{R}^N). \tag{1.8}$$

Given $F \in L^p(\Omega, \mathbb{R}^N)$, $1 < p$, a function $u \in W_0^{1,p}(\Omega)$ is a distributional solution to problem (1.1) if the equality

$$\int_\Omega \langle \mathcal{A}(x, \nabla u) + b(x, u), \nabla\varphi \rangle dx = \int_\Omega \langle F, \nabla\varphi \rangle \, dx \tag{1.9}$$

holds for every $\varphi \in W_0^{1,p'}(\Omega)$, where p' is the conjugate exponent to p, $pp' = p + p'$.

By the Sobolev Embedding Theorem in Lorentz-spaces [1], assumptions (1.2)–(1.7) ensure that (1.9) is meaningful when

$$E(x) \in L^{N,\infty}(\Omega), \tag{1.10}$$

(See Section 2 for more details). Observe that for $p < 2$ the energy functional

$$\int_\Omega \langle \mathcal{A}(x, \nabla u), \nabla u \rangle \, dx$$

need not to be bounded. In this case, we will refer to solutions as very weak solutions, introduced by Iwaniec and Sbordone in [2].

When $b \equiv 0$, an existence and uniqueness theorem is proved in [3,4] when $1 < p$ is close to two. In the linear case, $\mathcal{A}(x, \xi) = \mathcal{A}(x)\xi$ and $b(x, u) = b(x)u$ existence and uniqueness results for very weak solutions to problem (1.1) are obtained in [5].

We point out that under assumptions (1.2)–(1.8), the vector field

$$\mathscr{A}(x, s, \xi) = \mathcal{A}(x, \xi) + b(x, s)$$

satisfies the following conditions:

$$\langle \mathscr{A}(x, s, \xi), \xi \rangle \geqslant \alpha|\xi|^2 - E(x)|s||\xi| - |b_0(x)||\xi|$$
$$|\mathscr{A}(x, s, \xi)| \leqslant \beta|\xi| + E(x)|s| + |b_0(x)|$$

for any $\xi \in \mathbb{R}^N$, $s \in \mathbb{R}$ and a.e. $x \in \Omega$. Then, here we are dealing with a problem that is, in general, not coercive. Moreover, no classical structure conditions are satisfied (see [6]). In the linear case, it is classical to add a control on the norm of E [7,8] in order that the lower order term does not cause the lack of coercivity.

Without any bound on the norm of E, existence and uniqueness results are obtained in [9,4,5] by assuming E in L^N or in the Lorentz Space $L^{N,q}$, respectively.

On the other hand, as observed in [5], the only assumption E in $L^{N,\infty}$ does not guarantee the existence of a solution to problem (1.1). Anyway, as in the linear case the following uniqueness result holds in the large Sobolev space $W_0^{1,2)}$ (see Section 2 for definitions).

Theorem 1.1. *Assume that assumptions (1.2)–(1.5) and (1.7) are verified, and let*

$$E \in L^{N,\infty}(\Omega). \tag{1.11}$$

Then, there exists at most one solution $u \in W_0^{1,2)}(\Omega)$ of problem (1.1).

Under the stronger assumption that $E \in L^{N,q}(\Omega)$, we also obtain the following existence result.

Theorem 1.2. *Let assumptions (1.2)–(1.8) be verified and let*

$$E \in L^{N,q}(\Omega)$$

for $N \leqslant q < \infty$. Then, there exists $\delta_* = \delta_*(\alpha, \beta, N, r) > 0$ such that whenever

$$2 - \delta_* < p < 2 + \delta_*,$$

for any $F \in L^p(\Omega, \mathbb{R}^N)$ with $\operatorname{div} F \in L^1(\Omega)$, problem (1.1) admits a solution $u \in W_0^{1,p}(\Omega)$.

We point out that Theorem 1.2, when $2 < p < \delta_* + 2$, gives a regularity result for a solution to (1.1). When $p = 2$, we can assume in (1.8) $b_0 \in L^2(\Omega, \mathbb{R}^N)$ (see (5.25)). Moreover, we prove the following.