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The three-solutions theorem for p-Laplacian boundary value problems

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ABSTRACT

The three-solutions theorem is established for p-Laplacian boundary value problems in the presence of two pairs of lower and upper solutions, which are well ordered by the relationship between the lower and upper solutions and degree theory. In addition, an application of the three-solutions theorem is given.

1. Introduction

Consider the following p-Laplacian problem:

$$\begin{cases} (w(t)\varphi_p(u'(t)))' + \lambda f(t, u(t)) = 0, & \text{a.e. in } (0, 1), \\ u(0) = u(1) = 0, \end{cases} \tag{Q_λ}$$

where $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, λ is a positive parameter $f : (0, 1) \times \mathbb{R} \to \mathbb{R}$, $\mathbb{R} = (-\infty, \infty)$ and $w \in C([0, 1], \mathbb{R}_+)$, $\mathbb{R}_+ = [0, \infty)$.

Throughout this paper, we assume that the following conditions hold.

(F) $f : (0, 1) \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function, i.e.
- (i) for all $u \in \mathbb{R}$, $f(\cdot, u)$ is measurable;
- (ii) for a.e. $t \in [0, 1]$, $f(t, \cdot)$ is continuous;
- (iii) for every $M > 0$, there exists a function $h_M \in L^1(0, 1)$ such that, for a.e. $t \in [0, 1]$ and all $u \in \mathbb{R}$ with $|u| \le M$, we have $|f(t, u)| \le h_M(t)$.

(W_1) $w(t) > 0$ for $t \in (0, 1)$ and $\varphi_p^{-1}(1/w) \in L^1(0, 1)$.

Problem (Q_λ) arising from the study of radial solutions for the p-Laplacian boundary value problems on annular domains has received much attention in recent years. We refer the reader to [1–13] and the references therein. For example, Garaizar [5] studied the existence of one positive solution of (Q_λ) for small $\lambda > 0$, where $w(t) = t^{n-1}$ and $f(t, u) = t^{n-1}h(u)$, $h(0) < 0$, and $h(u) = O(u^k)$ for some $k > 1$.

Our interest in this paper is in establishing a three-solutions theorem (Theorem 3.2) for problem (Q_1). We will use the upper and lower solutions method associated with Leray–Schauder degree theory. The idea is credited to Amann [14] who proved his three-solutions theorem in the presence of two pairs of lower and upper solutions, which were well ordered, i.e., the lower solution was less than the upper one, by using the relationship between the lower and upper solutions and degree theory.

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By a solution u of problem (Q_1) we understand a function $u \in C[0,1] \cap C^1(0,1)$ with $w\varphi_p(u') \in AC[0,1]$ which satisfies (Q_1). One can easily see that solutions u of (Q_1) may not be in $C^1[0,1]$ if $w(t) = 0$ at $t = 0$ and/or 1. This causes difficulties in applying degree theory due to the lack of solution space regularity. To overcome this difficulty, we introduce a new space $C_w[0,1]$, which is analogous to $C^1[0,1]$, to get the desired result.

The existence of three solutions for nonlinear boundary value problems has received a lot of attention. We refer the reader to [15–27] and the references therein. For example, Ji and Ge [22] studied the multi-point boundary value problem with the nonlinearity depending on the first-order derivative. They provided sufficient conditions for the existence of at least three positive solutions by applying the Avery–Peterson fixed point theorem. Lee et al. [24] considered the steady state reaction–diffusion equation with nonlinearity having a falling zero. They showed the existence of three positive solutions for a certain range of λ by using an Amann type three-solutions theorem.

By applying our main result (Theorem 3.2), we give the existence result for three positive solutions of problem (Q_λ) for a certain range of λ under some assumptions imposed on the nonlinearity f (for details, see Section 4).

The rest of this paper is organized as follows. In Section 2, the operator corresponding to problem (Q_1) and the definitions which are essential for proving our results in this paper are introduced. In Section 3, we discuss the main result, the "three-solutions theorem". Finally, an application of our main result is given in Section 4.

2. Preliminaries

Consider the normed space

$$C_w[0,1] = \{u \in C[0,1] \cap C^1(0,1) : w^{1/(p-1)}u' \in C[0,1]\}$$

with norm

$$\|u\|_w = \|u\|_\infty + \|w^{1/(p-1)}u'\|_\infty,$$

where $\|u\|_\infty = \max_{t\in[0,1]} |u(t)|$. One can easily see that $(C_w[0,1], \|\cdot\|_w)$ is a Banach space; e.g., see Proposition 3.2 in [28].

Remark 2.1. Note that if $w(t) > 0$ for $t \in [0,1]$, then $(C_w[0,1], \|\cdot\|_w)$ is equivalent to $(C^1[0,1], \|\cdot\|_1)$. Here $\|u\|_1 = \|u\|_\infty + \|u'\|_\infty$.

As is well known, a solution operator plays a key role in degree theory. So we are concerned with a solution operator of the one-dimensional p-Laplacian problem

$$\begin{cases} (w(t)\varphi_p(u'(t)))' + K(t) = 0, & \text{a.e. in } (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{E_K}$$

where $K \in L^1(0,1)$.

By a solution u of problem (E_K), we understand a function $u \in C_w[0,1]$ with $w\varphi_p(u') \in AC[0,1]$ which satisfies (E_K). Problem (E_K) is equivalently written as

$$u(t) = G_p(K)(t) \triangleq \int_0^t \varphi_p^{-1}\left[\frac{1}{w(s)}\left(c(K) + \int_s^1 K(\tau)d\tau\right)\right] ds, \quad t \in [0,1],$$

where $c : L^1(0,1) \to \mathbb{R}$ is a mapping satisfying

$$\int_0^1 \varphi_p^{-1}\left[\frac{1}{w(s)}\left(c(K) + \int_s^1 K(\tau)d\tau\right)\right] ds = 0.$$

It can be proved that the mapping c is continuous and maps bounded sets in $L^1(0,1)$ into bounded sets in $\mathbb{R}$, and the mapping $G_p : L^1(0,1) \to C_w[0,1]$ is continuous and maps equi-integrable sets of $L^1(0,1)$ into relatively compact sets of $C_w[0,1]$ by arguments similar to those in the previous results (e.g., see [29–31]). Define $H : C_w[0,1] \to L^1(0,1)$ by $H(u)(t) = f(t, u(t))$. Then it is well known that H is a continuous operator which sends bounded sets of $C_w[0,1]$ into equi-integrable sets of $L^1(0,1)$. Thus $T = G_p \circ H : C_w[0,1] \to C_w[0,1]$ is completely continuous. Furthermore, (Q_1) has a solution u if and only if T has a fixed point u in $C_w[0,1]$.

Now, we give definitions which will be used in this paper.

Definition 2.2. Given functions $u, v, w : [0,1] \to \mathbb{R}$, we say that

(1) $u \le v$ if for all $t \in [0,1]$, $u(t) \le v(t)$.
(2) $u \in [v, w]$ if $v \le u \le w$.

Definition 2.3. Let $u, v \in C_w[0,1]$. We say that $u \prec v$ if and only if

(i) for all $t \in (0,1)$, $u(t) < v(t)$,
(ii) either $u(0) < v(0)$ or $(w^{1/(p-1)}u')(0) < (w^{1/(p-1)}v')(0)$,
(iii) either $u(1) < v(1)$ or $(w^{1/(p-1)}u')(1) > (w^{1/(p-1)}v')(1)$.