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# Elliptic systems involving the critical exponents and potentials

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ABSTRACT

In this paper, we investigate a singular elliptic system, which involves the critical Sobolev exponent and multiple Hardy-type terms. By employing variational methods, the existence of its positive solutions is established. By the Moser iteration method, some asymptotic properties of its nontrivial solutions at the singular points are verified.



## 1. Introduction

In this paper, we study the following elliptic system:

$$\begin{cases} -\Delta u - \dfrac{\mu_1 u}{|x-a_1|^2} = |u|^{2^*-2}u + \dfrac{\eta\alpha}{\alpha+\beta}|u|^{\alpha-2}|v|^{\beta}u + \lambda_1|u|^{q_1-2}u, & x \in \Omega, \\ -\Delta v - \dfrac{\mu_2 v}{|x-a_2|^2} = |v|^{2^*-2}v + \dfrac{\eta\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v + \lambda_2|v|^{q_2-2}v, & x \in \Omega, \\ u = v = 0, \quad x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$, $\eta > 0, a_i \in \Omega$, $\lambda_i > 0, \mu_i < \bar{\mu}, 2 \leq q_i < 2^*$, $i = 1, 2, \alpha,\ \beta > 1, \alpha+\beta = 2^*, 2^* := \frac{2N}{N-2}$ is the critical Sobolev exponent and $\bar{\mu} := \left(\frac{N-2}{2}\right)^2$ is the best Hardy constant.

We work in the product energy space $H \times H$, where $H := H_0^1(\Omega)$ denotes the completion of $C_0^\infty(\Omega)$ with respect to the norm $(\int_\Omega |\nabla u|^2\, dx)^{1/2}$. The corresponding energy functional of the problem (1.1) is defined in $H \times H$ by

$$J(u, v) := \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + |\nabla v|^2 - \frac{\mu_1 u^2}{|x-a_1|^2} - \frac{\mu_2 v^2}{|x-a_2|^2}\right) - \frac{\eta}{2^*}\int_\Omega |u|^\alpha |v|^\beta$$
$$- \frac{1}{2^*}\int_\Omega \left(|u|^{2^*} + |v|^{2^*}\right) - \frac{\lambda_1}{q_1}\int_\Omega |u|^{q_1} - \frac{\lambda_2}{q_2}\int_\Omega |v|^{q_2}. \tag{1.2}$$

Then $J \in C^1(H \times H, \mathbb{R})$. The duality product between $H \times H$ and its dual space $(H \times H)^{-1}$ is defined as

$$\langle J'(u, v),\ (\varphi, \phi)\rangle := \int_\Omega \left(\nabla u \nabla\varphi + \nabla v \nabla\phi - \frac{\mu_1 u\varphi}{|x-a_1|^2} - \frac{\mu_2 v\phi}{|x-a_2|^2}\right) dx$$

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$$- \int_\Omega \left( |u|^{2^*-2}u\varphi + |v|^{2^*-2}v\phi + \lambda_1|u|^{q_1-2}u\varphi + \lambda_2|v|^{q_2-2}v\phi \right) dx$$
$$- \int_\Omega \left( \frac{\eta\alpha}{\alpha+\beta}|u|^{\alpha-2}|v|^{\beta}u\varphi + \frac{\eta\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v\phi \right) dx,$$

where $u, v, \varphi, \phi \in H$ and $J'(u, v)$ denotes the Fréchet derivative of $J$ at $(u, v)$. A pair of functions $(u, v) \in H \times H$ is said to be a solution of the problem (1.1) if

$$\langle J'(u, v), (\varphi, \phi)\rangle = 0, \quad \forall(\varphi, \phi) \in H \times H.$$

The nontrivial solution of (1.1) is then a nonzero critical point of $J(u, v)$. By the standard elliptic argument we can verify that the solution $(u, v)$ of (1.1) has the following property:

$$u, v \in C^2(\Omega \setminus \{a_1, a_2\}) \cap C^1(\bar{\Omega} \setminus \{a_1, a_2\}). \tag{1.3}$$

The problem (1.1) is related to the well-known Hardy inequality [1,2]:

$$\int_{\mathbb{R}^N} \frac{|u|^2}{|x-a|^2} dx \le \frac{1}{\bar{\mu}} \int_\Omega |\nabla u|^2 dx, \quad \forall\ u \in C_0^\infty(\mathbb{R}^N),\ a \in \mathbb{R}^N.$$

By the Hardy inequality, the operator $L := (-\Delta \cdot -\mu\frac{\cdot}{|x-a|^2})$ is positive for any $\mu < \bar{\mu}$ and $a \in \Omega$. Therefore the first eigenvalue of $L$ is well defined:

$$\Lambda_1(\mu) := \inf_{u\in H\setminus\{0\}} \frac{\int_\Omega \left(|\nabla u|^2 - \frac{\mu u^2}{|x-a|^2}\right) dx}{\int_\Omega |u|^2 dx}, \quad \mu \in (-\infty,\ \bar{\mu}),\ a \in \Omega.$$

Furthermore, we can define the following best constant:

$$S(\mu) := \inf_{u\in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} \left(|\nabla u|^2 - \frac{\mu u^2}{|x-a|^2}\right) dx}{\left(\int_{\mathbb{R}^N} |u|^{2^*} dx\right)^{\frac{2}{2^*}}}, \quad \mu \in (-\infty,\ \bar{\mu}),\ a \in \mathbb{R}^N,$$

where the space $D^{1,2}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $(\int_{\mathbb{R}^N} |\nabla u|^2 dx)^{1/2}$. Note that $S(\mu)$ is independent of the point $a$ and $S(0) = S$, where $S$ is the well-known best Sobolev constant [3]. For $0 \le \mu < \bar{\mu}$, we infer from [4] that $S(\mu)$ is attained by the extremal function

$$V^\varepsilon_{a,\mu}(x) := \varepsilon^{\frac{2-N}{2}} U_\mu\left(\frac{x-a}{\varepsilon}\right), \quad \forall\varepsilon > 0, \tag{1.4}$$

where $U_\mu(x) = U_\mu(|x|)$ is a radially symmetric function and has the explicit form:

$$U_\mu(x) = \left(\frac{2N(\bar{\mu}-\mu)}{\sqrt{\bar{\mu}}}\right)^{\frac{\sqrt{\bar{\mu}}}{2}} \left(|x|^{\frac{\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu}}{\sqrt{\bar{\mu}}}} + |x|^{\frac{\sqrt{\bar{\mu}}+\sqrt{\bar{\mu}-\mu}}{\sqrt{\bar{\mu}}}}\right)^{-\sqrt{\bar{\mu}}}.$$

The function $V^\varepsilon_{a,\mu}(x)$ solves the equation

$$-\Delta u - \frac{\mu\, u}{|x-a|^2} = |u|^{2^*-2} u \quad \text{in } \mathbb{R}^N \setminus \{a\},$$

and satisfies

$$\int_{\mathbb{R}^N} \left( |\nabla V^\varepsilon_{a,\mu}(x)|^2 - \mu\frac{|V^\varepsilon_{a,\mu}(x)|^2}{|x-a|^2} \right) = \int_{\mathbb{R}^N} |V^\varepsilon_{a,\mu}(x)|^{2^*} = S(\mu)^{\frac{N}{2}}.$$

Furthermore,

$$S(\mu) = \left(1 - \frac{\mu}{\bar{\mu}}\right)^{\frac{N-1}{N}} S, \quad 0 \le \mu < \bar{\mu}.$$

If $\mu < 0$, from the result in [5] it follows that $S(\mu) = S(0) = S$.

On the other hand, for any $a_1, a_2 \in \mathbb{R}^N$, $\mu_1, \mu_2 < \bar{\mu}$, $\alpha, \beta > 1$, $\alpha + \beta = 2^*$, we can define the following best constant in the space $\mathcal{D} := (D^{1,2}(\mathbb{R}^N) \setminus \{0\})^2$:

$$S_{\alpha,\beta}(\mu_1, \mu_2) := \inf_{(u,v)\in\mathcal{D}} \frac{\int_{\mathbb{R}^N} \left(|\nabla u|^2 + |\nabla v|^2 - \frac{\mu_1 u^2}{|x-a_1|^2} - \frac{\mu_2 v^2}{|x-a_2|^2}\right) dx}{\left(\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta dx\right)^{\frac{2}{2^*}}}. \tag{1.5}$$