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Limit cycles of a Z_3-equivariant near-Hamiltonian system ☆

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ABSTRACT

This paper presents a study on the number of limit cycles of a Z_3-equivariant near-Hamiltonian system of degree 5 which is a perturbation of a cubic Hamiltonian system. By using the Melnikov method, we obtain 15 limit cycles. With degrees 3 and 4, we respectively get 4 and 10 limit cycles.

1. Introduction

In 1901, Hilbert [1] posed 23 mathematical problems of which the second part of the 16th one is to find the maximal number and relative positions of limit cycles of planar polynomial systems of degree n

$$\dot{x} = P_n(x, y), \qquad \dot{y} = Q_n(x, y).$$

There are many works on finding the maximal number of limit cycles and rasing the lower bound of Hilbert number $H(n)$ for general planar polynomial systems or for individual degree of systems. A detailed introduction and related literatures can be found in Li [2], Schlomiuk [3], Ilyashenko [4], and Han[5].

Many studies had been done for planar systems close to Hamiltonian systems, especially for quadratic and cubic systems (see [6–17]). The main results are on the number of limit cycles which appear near a center, a periodic annular or a homoclinic loop by perturbations. The first order Melnikov function which is called also an Abelian integral plays an important role in getting these results.

Consider an analytic planar system of the form

$$\dot{x} = f(x, y), \qquad \dot{y} = g(x, y). \tag{1.1}$$

We let $z = x + \mathrm{i}y,\ \bar{z} = x - \mathrm{i}y$ so that (1.1) becomes a system of complex equations below

$$\dot{z} = F(z, \bar{z}), \qquad \dot{\bar{z}} = \overline{F(z, \bar{z})}, \tag{1.2}$$

where $F(z, \bar{z}) = f(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}) + \mathrm{i}g(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i})$. If (1.2) is invariant under the rotation $\omega = \mathrm{e}^{\frac{2\pi}{q}i}z$, then we call (1.1) or (1.2) Z_q-equivariant.

According to [6,8], (1.1) or (1.2) is Z_q-equivariant if and only if the function F has the form

$$F(z, \bar{z}) = \sum_{k\geq 1} g_k(|z|^2)\bar{z}^{qk-1} + \sum_{k\geq 0} h_k(|z|^2)z^{qk+1},$$

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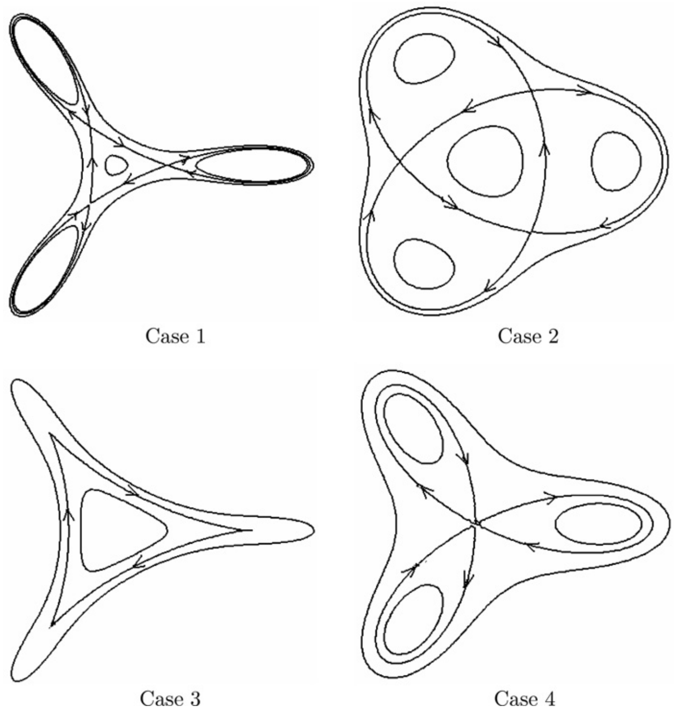

Fig. 1.1. The phase portraits of 4 cases of cubic Z_3-equivariant Hamilton systems.

where $g_k(r)$ and $h_k(r)$ are complex functions. When $f(x, y)$ and $g(x, y)$ in (1.1) satisfy

$$f(x, y) = H_y, \qquad g(x, y) = -H_x,$$

we call (1.1) Z_q-equivariant Hamilton system with the Hamilton function $H(x, y)$.

There have been some interesting studies on the bifurcation of limit cycles of equivariant polynomial systems. See Yu and Han [10,9,12], Wu [13,14]. In [9], it was shown that Z_2-equivariant cubic systems can have 12 small amplitude limit cycles. In [10], it was shown that Z_3-equivariant cubic systems can have 4 limit cycles. In this paper, we discuss the number of limit cycles for Z_3-equivariant polynomial systems with degrees 3, 4 and 5 by perturbing a cubic system, obtaining 4, 10, and 15 limit cycles respectively.

A cubic Z_3-equivariant system has the form

$$\begin{aligned} \dot{x} &= a_0x - b_0y + a_2x^2 + 2b_2xy - a_2y^2 + a_1x^3 - b_1x^2y + a_1xy^2 - b_1y^3, \\ \dot{y} &= b_0x + a_0y + b_2x^2 - 2a_2xy - b_2y^2 + b_1x^3 + a_1x^2y + b_1xy^2 + a_1y^3. \end{aligned} \tag{1.3}$$

The divergence of (1.3) is

$$\mathrm{div}(1.3) = 2a_0 + 4a_1(x^2 + y^2).$$

Thus, if $a_0 = a_1 = 0$, then (1.3) is a Z_3-equivariant Hamiltonian system.

From [8], we know that for cubic Z_3-equivariant Hamilton systems there are 4 cases below:

Case 1. $a_0 = a_1 = a_2 = 0,\ b_0b_2 < 0,\ 0 < 4b_0b_1 < b_2^2$;

Case 2. $a_0 = a_1 = a_2 = 0,\ b_0b_2 > 0,\ b_1b_2 < 0$;

Case 3. $a_0 = a_1 = a_2 = 0,\ b_0b_2 > 0,\ 4b_0b_1 = b_2^2$;

Case 4. $a_0 = a_1 = a_2 = b_0 = 0,\ b_1b_2 < 0$.

The phase portraits of these cases are given in Fig. 1.1. In this paper, we consider case 1 by taking $a_0 = a_1 = a_2 = 0,\ b_0 = -3,\ b_1 = -1,\ b_2 = 4$. Then the perturbed system is taken to have the form

$$\begin{aligned} \dot{x}_1 &= 3y_1 + 8x_1y_1 + x_1^2y_1 + y_1^3 + \varepsilon p^*(x_1, y_1), \\ \dot{y}_1 &= -3x_1 + 4x_1^2 - 4y_1^2 - x_1^3 - x_1y_1^2 + \varepsilon q^*(x_1, y_1), \end{aligned} \tag{1.4}$$

where $p^*(x_1, y_1)$, $q^*(x_1, y_1)$ are polynomials of degree 5, such that (1.4) is Z_3-equivariant.