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Bad normed spaces, convexity properties, separated sets

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ABSTRACT

Some years ago, a parameter – denoted by $A_1(X)$ – was defined in real Banach spaces. In the same setting, several years before, a notion called Q-*convexity* had been defined. Studying these two notions seems to be rather awkward and up until now this has not been done in deep.

Here we indicate some properties and connections between these two parameters and some other related ones, in infinite-dimensional Banach spaces. We also consider another notion, a natural extension of Q-convexity, and we discuss the case when $A_1(X)$ attains its maximum value. The spaces where this happens can be considered as "bad" since they cannot have several properties which are usually considered as nice (like uniform non-squareness or P-convexity).

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1. Introduction and definitions

Let X be an infinite-dimensional, real Banach space.
We shall use the following notations:

$S_X = \{x \in X : \|x\| = 1\}$;
$B_X = \{x \in X : \|x\| \leq 1\}$.

The following parameter was introduced in [1] (see also [2–4]):

$$A_1(X) = \frac{1}{2} \inf_{x \in S_X} \left\{ \sup_{y \in S_X} \{\|x+y\| + \|x-y\|\} \right\}. \tag{1}$$

The following notion was defined in [5], p. 286, and did not receive attention later.
Given $\varepsilon > 0$ and $n \geq 2$, say that X is $Q(n, \varepsilon)$-*convex* if for no $x_1, x_2, \ldots, x_n \in B_X$ we have:

$$\|x_1 + \cdots + x_{k-1} - x_k\| > k - \varepsilon \quad \text{for } k = 2, \ldots, n. \tag{2}$$

A space X is called Q-*convex* if it is $Q(n, \varepsilon)$-convex for some $n \in N$, $\varepsilon > 0$.

We generalize the last notion in the following way; clearly next property is implied by Q-convexity.

Definition. We say that X is Q_∞-*convex* if for some $\varepsilon > 0$ we cannot find a sequence $(x_n)_{n \in N}$ in B_X such that

$$\left\| \sum_{i=1}^{n-1} x_i - x_n \right\| > n - \varepsilon \quad \text{for every } n \in N. \tag{3}$$

In this paper we want to study the relations among Q-*convexity*, the condition $A_1(X) < 2$ and some other properties.

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Note that several other similar properties were introduced and studied in the paper [5]; some of them had already been considered in [6]. We recall in passing, that in the picture at p. 289 of [5], E-convexity and F-convexity are exchanged.

A new interest on some of these properties arose recently because some of them imply the fixed point property for non-expansive mappings: see [7,8].

Recall that non-expansive mappings are those which have a Lipschitz constant equal to one. A Banach space is said to have the weak fixed point property (WFPP), whenever every non-expansive self-mapping of any nonempty weakly compact convex subset K of X has a fixed point. If the same holds for every nonempty closed convex bounded subset K of X it is said that X has the fixed point property (FPP) for short. For reflexive spaces it is obvious that the two properties coincide.
On the other hand, ℓ_1 and c_0 enjoy (WFPP) but lack of (FPP).

Next Section 2 contains the main result. Some consequences of it are given in Section 3, where other constants are considered. Finally, in Section 4 we indicate some further properties concerning $A_1(X)$.

2. Main result

In this section we shall prove our main result.
Before proving it, we indicate a remark, and a lemma concerning the above definitions.

Remark 1. It is simple to see that in the definitions of Q-convexity and Q_∞-convexity, we may replace the condition(s) $x_1, x_2, \ldots, x_n (\ldots) \in B_X$, with the same $\in S_X$: this can be seen by observing that in (3) all the elements x_n must have a norm larger than $1-\varepsilon$.

Lemma 2 *([5], Proposition 3.2; or also, but with 2ε Instead of ε, [9], p. 102)***.** *Assume that a sequence $(x_n)_{n\in N}$ satisfies (3); then it also satisfies:*

$$\|x_n - y\| > 2-\varepsilon \text{ whenever } y \text{ is a convex combination of } x_1, \ldots, x_{n-1}. \tag{4}$$

To prove our main result we shall use the following.

Fact. The infinite product $\prod_{n=1}^{\infty}(1+\frac{1}{n^2})$ converges. We shall denote by a its value.

Theorem 3. *If X is Q_∞-convex (in particular: if X is Q-convex), then $A_1(X) < 2$.*

Proof. Let $A_1(X) = 2$: this means that given $\varepsilon > 0$, if $x \in S_X$, we can always find $y \in S_X$ such that

$$\|x-y\| + \|x+y\| > 4-\varepsilon;$$

this implies

$$\min\{\|x-y\|,\ \|x+y\|\} > 2-\varepsilon. \tag{5}$$

Let $\varepsilon \in (0,1)$ be given; set $\varepsilon_2 = \frac{2\varepsilon}{a}$, then let $\|x_1 \mp x_2\| > 2-\varepsilon_2$.

We want to show that we can find a sequence $x_1, x_2, \ldots, x_n, \ldots$ in S_X such that (for $n \geq 2$):

$$\|x_1 + x_2 + \cdots + x_{n-1} \mp x_n\| > n - \varepsilon_n, \quad \text{with } \varepsilon_n = \frac{\varepsilon_2}{2}\prod_{i=1}^{n-1}\left(1+\frac{1}{i^2}\right). \tag{6}$$

Note that $\varepsilon_n < \frac{\varepsilon_2 a}{2} = \varepsilon$ for every n.

We reason by induction. Suppose that $x_1, x_2, \ldots, x_n$ $(n \geq 2)$ satisfying (6) for $\varepsilon_n \in (0,1)$ as above have already been found. Set

$$\|x_1 + x_2 + \cdots + x_n\| = c\ (c > n - \varepsilon_n > n-\varepsilon > 1). \tag{7}$$

We can find $x_{n+1} \in S_X$ such that $\|\frac{x_1+x_2+\cdots+x_n}{c} \mp x_{n+1}\| > 2 - \frac{\varepsilon_n}{c\,n^2}$. This implies

$$\begin{aligned}
\|x_1 + x_2 + \cdots + x_n \mp x_{n+1}\| &= c\left\|\frac{x_1+x_2+\cdots+x_n}{c} \mp \frac{x_{n+1}}{c}\right\| \\
&\geq c\left(\left\|\frac{x_1+x_2+\cdots+x_n}{c} \mp x_{n+1}\right\| - \left\|\mp x_{n+1} \pm \frac{x_{n+1}}{c}\right\|\right) \geq c\left(2 - \frac{\varepsilon_n}{c\,n^2} - \left|1-\frac{1}{c}\right|\right) \\
&= c\left(1+\frac{1}{c}-\frac{\varepsilon_n}{c\,n^2}\right) = c+1-\frac{\varepsilon_n}{n^2} > n-\varepsilon_n+1-\frac{\varepsilon_n}{n^2} = n+1-\varepsilon_{n+1}.
\end{aligned}$$

Thus we can find $x_1, x_2, \ldots, x_{n+1}$ that still satisfy (6). Moreover, since $\varepsilon_n < \varepsilon$ for every n, we see that (6) implies (3). This concludes the proof. □

Remark 4. It is not difficult to see that the condition $A_1 < 2$ does not imply Q_∞-convexity. For example, let $X = \ell_\infty$: we have $A_1(\ell_\infty) = 3/2$, but ℓ_∞ is not Q_∞-convex.

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