Nonlinear Analysis 73 (2010) 1515–1532

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Existence theorems for solutions to random fuzzy differential equations

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ARTICLE INFO

Article history:
Received 27 May 2009
Accepted 24 April 2010

MSC:
34F05
34G20
34A12
26E50

Keywords:
Random fuzzy differential equations
Initial value problem for fuzzy mappings
Fuzzy derivative
Fuzzy stochastic process
Fuzzy random variable

ABSTRACT

We examine a random fuzzy initial value problem with two kinds of fuzzy derivatives. For both cases we establish the results of existence and uniqueness of local solutions to random fuzzy differential equations. The existence of global solutions is also obtained.

1. Introduction

Fuzzy differential equations (FDEs) is a topic very important from the theoretical point of view (see e.g. [1,2] and the references therein) as well as of their applications, for example, in civil engineering [3], in modeling hydraulic [4] and in population models [5]. There are many suggestions on how to define a fuzzy derivative and consequently several ways to study FDEs. Various types of derivatives of fuzzy functions were compared, and the solutions of FDEs related to them were investigated in [6]. Fuzzy differential equations were first formulated by Kaleva [7,8]. He used the concept of H-differentiability which was introduced by Puri and Ralescu [9], and obtained the existence and uniqueness theorem for a solution of FDE under the Lipschitz condition, whereas in [8] he characterized those subsets of the fuzzy set space in which the Peano theorem is valid. Since then there appeared a lot of papers concerning the theory and applications of fuzzy differential equations, fuzzy dynamics and fuzzy differential inclusions (see e.g. [10,11,5,12–30]). For a significant collection of results from the theory of FDEs we refer to the monograph of Lakshmikantham and Mohapatra [1], and to Diamond and Kloeden [2].

In this paper we will consider random fuzzy differential equations (RFDEs) as they can provide good models of dynamics of real phenomena which are subjected to two kinds of uncertainties: randomness and fuzziness, simultaneously. The first source of uncertainty is connected with the uncertainty in prediction of the outcome of an experiment. Randomness intends to break the law of causality and the probabilistic methods are applied in its analysis. Fuzziness means nonstatistical inexactness that is due to subjectivity and imprecision of human knowledge rather than to the occurrence of random events. It is caused by the lack of sharply defined criteria of membership in the sets of some considered space (a simple example could be a class of all real numbers which are much greater than 1). Fuzziness intends to break the law of excluded middle and is appropriately treated by fuzzy set theory. The probability and fuzzy set theories team up in the concept of fuzzy random variable. This notion is a crucial one in the analysis of RFDEs.

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0362-546X/$ – see front matter
doi:10.1016/j.na.2010.04.049

In the literature one can find various definitions of fuzzy random variables. For the first time the concept of fuzzy random variable was proposed by Kwakernaak [31]. Further it was used by Kruse and Meyer [32]. In [33–35,2] there appear two notions of measurability of fuzzy mappings. The relations between different concepts of measurability for fuzzy random variables are contained in the papers of Colubi et al. [36], Terán Agraz [37], López-Diaz and Ralescu [38]. In this paper we will use a definition of fuzzy random variable which was introduced by Puri and Ralescu [35]. This definition is currently the most often used in probabilistic and statistical aspects of the theory of fuzzy random variables.

In papers [39–41] one can find the studies of differential equations where the two kinds of uncertainties are incorporated. Feng [40] considered fuzzy stochastic differential systems using a notion of mean-square derivative (which is different from our derivative) and mean-square integral of second-order fuzzy stochastic processes introduced by himself in [42]. The fuzzy stochastic process is viewed there as a mapping acting from an interval to the space of second-order fuzzy random variables. In his setting, the fuzzy random variable comes from a narrower class than ours. The existence and uniqueness of a Cauchy problem is then obtained under an assumption that the coefficients satisfy a condition with the Lipschitz constant. The proof is based on the application of the Banach fixed point theorem. In [39] the existence and uniqueness of the solution for RFDEs with non-Lipschitz coefficients is proven. The values of fuzzy mappings are in the space of fuzzy sets of a reflexive separable Banach space. However only the autonomous case is treated, where right-hand side is non-random and initial value is a constant fuzzy set. The behaviour of solutions to the Cauchy problem such as existence, uniqueness, lifetime, dependence on initial values and non-confluence is studied. The author uses a concept of the support function of a fuzzy set which allows him to consider the random scalar differential equations instead of RFDEs. The main idea in the proofs is to construct a family of positive increasing functions so that the Gronwall lemma can be applied to the composition of these functions with appropriate processes. Malinowski [41] considered RFDEs with fuzzy derivative defined as in [9]. The coefficients of the equation were random fuzzy functions, also the initial condition was treated as a fuzzy random variable. With an assumption that the right-hand side of the equation satisfies a global Lipschitz-type condition the existence and uniqueness of the solution to RFDEs was proven.

Random fuzzy initial value problem can be viewed as non-random one but with a parameter. However the rich theory of non-random FDEs cannot be straightforward applied. As the examples show (see [41]), the existence of solution to the deterministic version of random fuzzy initial value problem does not determine the existence of a random solution. A solution to RFDE is a fuzzy stochastic process, i.e. a family of fuzzy random variables. Therefore in the proving of the existence of a solution we apply the method of successive approximations as opposed to the non-random case where the method of fixed point is frequently used.

We examine RFDEs with two kinds of fuzzy derivatives and obtain parallel results for both settings. Supposing that the Lipschitz condition holds on bounded sets, we establish the existence and uniqueness of a local solution to RFDEs. The existence of at least one local solution is obtained under assumption that the right-hand side of the equation satisfies some integrability condition. This result is then applied in a demonstration of existence of a global solution to RFDEs.

The paper is organized as follows. In Section 2 we collect the fundamental notions and facts which will be used in the rest of the article, the formulation of the main problem is also contained. In Section 3 we discuss RFDEs where the fuzzy derivative is understood in the sense of Puri and Ralescu [9]. The theorems on existence of local and global solutions are presented. Section 4 is a parallel one to the Section 3. We consider there RFDEs with second type of fuzzy derivative which was proposed in [43,13]. Theorems similar to those in Section 3 are stated. In Section 5 we present some examples which illustrate the theory of RFDEs.

2. Preliminaries

In this section our aim is to give a background of the fuzzy set space, and an overview of properties used by us of integration and differentiation of fuzzy set-valued mappings.

Let A, B be nonempty compact subsets of $\mathbb{R}^d$. The Hausdorff metric is defined as follows

$$d_H(A, B) = \max \left\{ d_H^*(A, B), d_H^*(B, A) \right\},$$

where $d_H^*(A, B) = \sup_{x \in A} \inf_{y \in B} \|x - y\|$, and $\|\cdot\|$ denotes usual Euclidean norm in $\mathbb{R}^d$.

We have $d_H^*(A, B) = 0$ if and only if $A \subset B$, and $d_H^*(A, B) \le d_H^*(A, C) + d_H^*(C, B)$ for nonempty compact subsets A, B, C of $\mathbb{R}^d$.

Let $\mathcal{K}(\mathbb{R}^d)$ denote a family of all nonempty compact convex subsets of $\mathbb{R}^d$ and define addition and scalar multiplication in $\mathcal{K}(\mathbb{R}^d)$ as usual, i.e. for $A, B \in \mathcal{K}(\mathbb{R}^d)$ and $\lambda \in \mathbb{R}$

$$A + B = \{a + b \mid a \in A, b \in B\}, \qquad \lambda A = \{\lambda a \mid a \in A\}.$$

Denote

$$E^d = \{u : \mathbb{R}^d \to [0, 1] \mid u \text{ satisfies (i)–(iv) below}\},$$

(i) u is normal, i.e. there exists $x_0 \in \mathbb{R}^d$ such that $u(x_0) = 1$,
(ii) u is fuzzy convex, i.e. $u(\lambda x + (1 - \lambda)y) \ge \min\{u(x), u(y)\}$ for any $x, y \in \mathbb{R}^d$ and $\lambda \in [0, 1]$,
(iii) u is upper semicontinuous,
(iv) $[u]^0 = \mathrm{cl}\{x \in \mathbb{R}^d \mid u(x) > 0\}$ is compact, where cl denotes the closure in $(\mathbb{R}^d, \|\cdot\|)$.