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Global classical solutions of mixed initial-boundary value problem for quasilinear hyperbolic systems

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ABSTRACT

In this paper, we consider the mixed initial-boundary value problem for quasilinear hyperbolic systems with nonlinear boundary conditions in the first quadrant $\{(t, x)| t \geq 0, x \geq 0\}$. Under the assumptions that the system is strictly hyperbolic and linearly degenerate or weakly linearly degenerate, the global existence and uniqueness of C^1 solutions are obtained for small initial and boundary data. We also present two applications for physical models.

1. Introduction and main results

Consider the following first order quasilinear strictly hyperbolic system

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = 0, \tag{1.1}$$

where $u = (u_1, \ldots, u_n)^T$ is the unknown vector function of (t, x) and $A(u)$ is an $n \times n$ matrix with suitably smooth elements $a_{ij}(u)$ $(i, j = 1, \ldots, n)$.

By the definition of strict hyperbolicity, for any given u on the domain under consideration, $A(u)$ has n distinct real eigenvalues $\lambda_1(u), \ldots, \lambda_n(u)$. We furthermore suppose that

$$\lambda_1(0) < \cdots < \lambda_m(0) < 0 < \lambda_{m+1}(0) < \cdots < \lambda_n(0). \tag{1.2}$$

Let $l_i(u) = (l_{i1}(u), \ldots, l_{in}(u))$ (resp., $r_i(u) = (r_{i1}(u), \ldots, r_{in}(u))^T$) be a left (resp., right) eigenvector corresponding to $\lambda_i(u)$ $(i = 1, \ldots, n)$:

$$l_i(u)A(u) = \lambda_i(u)l_i(u) \ (\text{resp., } A(u)r_i(u) = \lambda_i(u)r_i(u)). \tag{1.3}$$

We have

$$\det|l_{ij}(u)| \neq 0 \ (\text{equivalently, } \det|r_{ij}(u)| \neq 0). \tag{1.4}$$

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Without loss of generality, we suppose that

$$l_i(u)r_j(u) \equiv \delta_{ij} \quad (i, j = 1, \ldots, n), \tag{1.5}$$

$$r_i(u)^T r_i(u) \equiv 1 \quad (i = 1, \ldots, n), \tag{1.6}$$

where δ_{ij} stands for Kronecker's symbol.

All $\lambda_i(u)$, $l_{ij}(u)$ and $r_{ij}(u)$ $(i, j = 1, \ldots, n)$ are supposed to have the same regularity as $a_{ij}(u)$ $(i, j = 1, \ldots, n)$.

For the Cauchy problem of system (1.1) with the initial data

$$t = 0 : u = \phi(x), \quad x \in \mathbb{R}, \tag{1.7}$$

where $\phi(x)$ is a C^1 vector function with bounded C^1 norm, many results have been obtained for the global existence of classical solutions (see [1–4]). In particular, by means of the concept of weak linear degeneracy, for small initial data with certain decaying properties, the global existence and the blow-up phenomenon of C^1 solution to Cauchy problem (1.1) and (1.7) have been completely studied (see [5–11], also see [12–14]). In terms of two basic L^1 estimates, Zhou [15] furthermore relaxed the limitations on initial data and then showed that Cauchy problem (1.1) and (1.7) for a weakly linearly degenerate and strictly hyperbolic system admits a unique global C^1 solution which also satisfies L^1 stability.

In order to consider the effect of nonlinear boundary conditions on the global regularity of classical solution of system (1.1), Li & Wang [16] investigated the mixed initial-boundary value problem for system (1.1) on a half-unbounded domain. The result obtained by Li & Wang indicates that the interaction of nonlinear boundary conditions with nonlinear hyperbolic waves does not cause any negative effect on the global regularity of the C^1 solution, provided that the C^1 norms of initial and boundary data both decaying at infinity are small enough. In this paper, we are going to reprove the global existence result with less restrictions on the initial and boundary data. In particular, the supreme norms of the derivatives of the initial and boundary data are not assumed to be small.

Moreover, for the mixed initial-boundary value problem on a bounded domain $\{(t, x) \mid t \geq 0,\ 0 \leq x \leq L\}$, the results on the global regularity can be found in [17,7,18–20].

On the domain

$$D := \{(t, x) | \, t \geq 0,\ x \geq 0\}, \tag{1.8}$$

we consider the mixed initial-boundary value problem for system (1.1) with the initial data

$$t = 0 : \quad u = \phi(x), \quad x \geq 0 \tag{1.9}$$

and the boundary condition

$$x = 0 : v_s = f_s(\alpha(t), v_1, \ldots, v_m) + h_s(t) \quad (s = m + 1, \ldots, n) \tag{1.10}$$

where

$$v_i = l_i(u)u \quad (i = 1, \ldots, n), \tag{1.11}$$

$h_s(t)$ $(s = m + 1, \ldots, n)$ are given C^1 functions of t, and

$$\alpha(t) := (\alpha_1(t), \ldots, \alpha_k(t)).$$

Let

$$h(t) := (h_{m+1}(t), \ldots, h_n(t)),$$

and we define

$$|u| := \left(\sum_{k=1}^{n} u_k^2 \right)^{\frac{1}{2}}$$

for any vector value function $u = (u_1, \ldots, u_n)^T$. Without loss of generality, we suppose that

$$f_s(\alpha(t), 0, \ldots, 0) \equiv 0 \quad (s = m + 1, \ldots, n). \tag{1.12}$$

We remark that, in a neighborhood of $u = 0$, the boundary condition (1.10) takes the same form under any possibly different choice of left eigenvectors.

To state our results precisely, we shall first recall the concept of linear degeneracy and weak linear degeneracy (see [7] or [10]) as follows.

Definition 1.1. For any given u on the domain under consideration, the i-th characteristic $\lambda_i(u)$ is called linearly degenerate in the sense of P.D. Lax, provided that

$$\nabla \lambda_i(u) r_i(u) \equiv 0. \tag{1.13}$$

If all characteristics $\lambda_i(u)$ $(i = 1, \ldots, n)$ are linearly degenerate, then system (1.1) is referred to as linearly degenerate.