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The existence of large solutions of semilinear elliptic equations with negative exponents ☆

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ABSTRACT

In this work, we consider semilinear elliptic equations with boundary blow-up whose nonlinearities involve a negative exponent. Combining sub- and super-solution arguments, comparison principles and topological degree theory, we establish the existence of large solutions. Furthermore, we show the existence of a maximal large positive solution.

1. Introduction

In this paper we study the existence of positive solutions and the maximal positive solution of elliptic equations with boundary blow-up

$$\begin{cases} -\Delta u = a(x)u^{-m} - b(x)u^p, & x \in \Omega, \\ u = +\infty, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded and smooth domain, and the function $b \in C^\eta(\bar{\Omega})$ is a positive function. Throughout this paper, we always suppose $p > 1, m > 0, a \in C^\eta(\bar{\Omega})$.

In (1.1), $u = +\infty$ on $\partial\Omega$ means that $u(x) \to +\infty$ as $x \to \partial\Omega$. Generally, the solutions of (1.1) are often said to be large solutions. Problems related to boundary blow-up have been studied for a long time. Motivated by a geometric problem, Bieberbach in [1], for the cases $N = 2$ and $f(u) = e^u$, proved that the following problem has a unique solution $u \in C^2(\Omega)$

$$\begin{cases} -\Delta u = f(u), & x \in \Omega, \\ u = +\infty, & x \in \partial\Omega. \end{cases}$$

So far, there is a large amount of literature on elliptic equations related with boundary blow-up, apart from the following mentioned papers, the reader can also refer to [1–15]. In [16], Du introduced systematically

$$\begin{cases} -\Delta u = a(x)u - b(x)u^p, & x \in \Omega, \\ u = +\infty, & x \in \partial\Omega \end{cases}$$

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and gave some brilliant results involving the existence, uniqueness and asymptotic behaviour of positive solutions. In [17], Delgado, López-Gómez and Suárez studied the following problem

$$\begin{cases} -\Delta u = \lambda u^{1/m} - a(x)u^{p/m}, & x \in \Omega, \\ u = +\infty, & x \in \partial\Omega, \end{cases}$$

where Ω is a bounded and smooth domain and $1 < m < p$, $a(x) \geq 0$. Simultaneously, in [17] the open set $\Omega_+ := \{x \in \Omega : a(x) > 0\}$ is connected with boundary $\partial\Omega_+$ of class C^2, and the open set $\Omega_0 := \Omega \setminus \bar{\Omega}_+$ satisfies $\overline{\Omega}_0 \subset \Omega$. In [18], Delgado, López-Gómez and Suárez studied the existence of positive solutions of the problem

$$\begin{cases} -\Delta u = W(x)u^q - a(x)f(u), & x \in \Omega, \\ u = +\infty, & x \in \partial\Omega, \end{cases}$$

with $0 < q < 1$, for a rather general class of functions $f(u)$ (see Theorem 1.1 in [18]).

Few papers involving boundary blow-up deal with such nonlinearities with negative exponents. Since nonlinearities with negative exponents are singular at 0, 0 cannot be a sub-solution of the corresponding equations. Therefore, if $a(x)$ is a negative function or sign-changing function, we have difficulty in finding a sub-solution of the auxiliary problem

$$\begin{cases} -\Delta u = a(x)u^{-m} - b(x)u^p, & x \in \Omega, \\ u = \phi, & x \in \partial\Omega, \end{cases}$$

where ϕ is a positive function in $C^{1+\alpha}(\partial\Omega)$. In order to overcome the difficulty, we will show the existence of positive solutions of the following auxiliary problem by using topological degree theory under some conditions, and show the uniqueness of solutions which are larger than a constant γ

$$\begin{cases} -\Delta u = \lambda u^{-m} - \mu u^p, & x \in \Omega, \\ u = l, & x \in \partial\Omega, \end{cases}$$

where $-\lambda$, μ are positive constants.

In this paper, we also need the following lemma for operators involving the Laplacian, which goes back to [19,20].

Lemma 1. *Assume that $\Omega \subset R^N$ is a bounded domain of class $C^{2+\eta}$, $0 < \eta < 1$, and suppose that $a(x) \in C^{\eta}(\bar{\Omega})$. Then the following assertions are equivalent:*

(i) *$\mathscr{L} = -\Delta + a$ satisfies the maximum principle,*
(ii) *$\mathscr{L} = -\Delta + a$ satisfies the strong maximum principle,*
(iii) *$\sigma_1[\Delta + a, \Omega] > 0$, where $\sigma_1[\Delta + a, \Omega]$ denotes the first eigenvalue of $\mathscr{L}$ with a homogeneous Dirichlet boundary condition,*
(iv) *there exists a positive strict super-solution $\phi \in C^2(\Omega) \cap C^1(\bar{\Omega})$ such that $-\Delta\phi + a(x)\phi \geq 0$ and $\phi|_{\partial\Omega} \geq 0$ and ϕ is not a solution.*

In Section 2, we will give the main results involving the existence of positive solutions and the maximal positive solution of (1.1).

2. Existence of positive solutions of (1.1)

We need to give some necessary results. Denote $\gamma = \left(\frac{m}{p}\right)^{1/(m+p)}$. Firstly, we show that the following problem has a unique positive solution w_ϕ satisfying $w_\phi > \gamma$

$$\begin{cases} -\Delta u = \lambda u^{-m} - b(x)u^p, & x \in \Omega, \\ u = \phi, & x \in \partial\Omega, \end{cases} \tag{2.1}$$

where $\lambda \in \mathbb{R}$, $\phi \in C^{2+\eta}(\partial\Omega)$, $\phi(x) > 0$ on $\partial\Omega$, $m > 0$ and $p > 1$. The following proposition plays an important role in order to prove the existence of positive solutions of (2.1).

Proposition 1. *There exist two positive constants μ_0 and L_0 such that for any $-\lambda \leq \mu \leq \mu_0$ (where $\lambda < 0$) and $l \geq L_0$, the following problem has a unique positive solution w_l satisfying $w_l > \gamma$*

$$\begin{cases} -\Delta u = \lambda u^{-m} - \mu u^p, & x \in \Omega, \\ u = l, & x \in \partial\Omega. \end{cases} \tag{2.2}$$

Proof. Due to the strong maximum principle, it holds that any positive solution of (2.2) is between 0 and l. Denote $v = l - u$. For the existence of positive solutions of (2.2), we only need to investigate the following problem

$$\begin{cases} -\Delta v = \mu(l - v)^p - \lambda(l - v)^{-m}, & x \in \Omega, \\ v = 0, & x \in \partial\Omega. \end{cases} \tag{2.3}$$

Denote $C_0(\bar{\Omega}) = \{u \in C(\bar{\Omega}) : u|_{\partial\Omega} = 0\}$ and $D_{l,\gamma} = \{C_0(\bar{\Omega}) : 0 \leq v < l - \gamma\}$. Then, $D_{l,\gamma}$ is a bounded and relatively open subset of K, where $K = \{u \in C_0(\bar{\Omega}) : u \geq 0\}$ is the naturally positive cone of $C_0(\bar{\Omega})$. Define a map by

$$\mathscr{N}(t, w) = (-\Delta)^{-1}[\mu(l - w)^p - \lambda t(l - w)^{-m}].$$