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Existence of solutions to quasi-linear elliptic problems with generalized Robin boundary conditions

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ABSTRACT

We characterize the L^q-solvability of a class of quasi-linear elliptic equations involving the p-Laplace operator with generalized nonlinear Robin type boundary conditions on bad domains. Some uniqueness results are also given.

1. Introduction

Let $p \in (1, \infty)$ and let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded domain which has the $W^{1,p}$-extension property (see Definition 2.4 below). The goal of this paper is to establish necessary and sufficient conditions for the existence and uniqueness of solutions to the quasi-linear elliptic boundary value problem formally given by

$$\begin{cases} -\Delta_p u + \alpha(u) = F_1 & \text{in } \Omega, \\ \mathrm{dN}_p(u) + \beta(\mathrm{Tr}u)\mathrm{d}\mu = F_2 \mathrm{d}\mu & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

The measure μ is supposed to be an upper d-Ahlfors measure on the boundary $\partial\Omega$ for some $d \in (N-p, N) \cap (0, N)$. The distribution $\mathrm{dN}_p(u)$ denotes the *p-generalized normal derivative* (see Definition 2.1 below) of the function u, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplace operator, $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ are continuous and monotone nondecreasing functions and F_1, F_2 are given functions in $L^{q_1}(\Omega)$ and $L^{q_2}(\partial\Omega, \mu)$, respectively, for some $q_1, q_2 \in [1, \infty]$. They are two main difficulties in studying the above problem. The first difficulty is the nonlinearity and the second difficulty is the domain Ω since we allow domains with a strange geometry where the existence of a trace for functions in Sobolev spaces is not always easy to establish.

Several authors have considered this type of problem. The linear heat equation involving the operator defined in (1.1) (that is, $p = 2$, $F_2 = \alpha \equiv 0$ and $\beta(t) = t$) has been considered in [1] on arbitrary domains. The case $\mu = \sigma$, $F_2 \equiv 0$ (where σ denotes the restriction to $\partial\Omega$ of the $(N-1)$-dimensional Hausdorff measure $\mathcal{H}^{N-1}$) has been investigated by

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Daners and Drábek [2] where they have obtained some regularity results for weak solutions, but they have not investigated the existence of weak solutions. Recently, under additional restrictions on Ω, q_1, q_2, α and β, it has been shown in [3] that weak solutions of (1.1) belong to $L^\infty(\Omega)$. In the same paper, the authors have used some properties of reflexive Orlicz spaces to obtain some sufficient conditions (only in terms of q_1 and q_2) for the existence of weak solutions.

Before giving the main results of the paper, we briefly describe the motivation of our formulation of generalized Robin boundary conditions. First, note that the Robin boundary condition has many physical interpretations, for example it is a general form of the insulating boundary condition for convection–diffusion equations. It is also the boundary condition that usually derives in one physical domain from another in the case of problems like fluid-structure interaction models for aero-elasticity of airplanes or for aero-thermo-mechanics of aeronautical engines involving strong coupling. Next, given a measurable function $\beta \in L^\infty(\partial\Omega, \sigma)$, the linear Robin boundary condition (the case $p = 2$) is defined by investigating whenever the bilinear form

$$a_\beta(u, v) := \int_\Omega \nabla u \nabla v \mathrm{d}x + \int_{\partial\Omega} \beta u, v \mathrm{d}\sigma, \qquad D(a_\beta) := \left\{ u \in H^1(\Omega) \cap C(\overline{\Omega}), \int_{\partial\Omega} \beta |u|^2 \mathrm{d}\sigma < \infty \right\}$$

is closable in $L^2(\Omega)$. When it is closable, then the operator associated with its closure is the Robin Laplace operator Δ_R in $L^2(\Omega)$. If a_β is not closable (this may happen if Ω has a strange geometry), then it has a closable part and Δ_R is defined to be the operator associated with the closure of this closable part. If Ω is a fractal type set, like the von Koch snowflake, then $\mathcal{H}^{N-1}$ is locally infinite on $\partial\Omega$. Consequently, $D(a_\beta) = \{u \in H^1(\Omega) \cap C(\overline{\Omega}),\ u = 0 \text{ on } \partial\Omega\}$ and the operator Δ_R coincides with a realization of the Laplace operator Δ_D with the Dirichlet boundary condition. The same situation occurs when $\partial\Omega$ is an s-set with $s \in (N-1, N]$. This shows that in the cases where $\partial\Omega$ is an s-set, the natural measure on the boundary should be the s-dimensional Hausdorff measure $\mathcal{H}^s$ on $\partial\Omega$ where s denotes the Hausdorff dimension of $\partial\Omega$. For general $p \in (1, \infty)$, a realization of the p-Laplace operator Δ_p with the Robin boundary condition is interpreted similarly and one also has the same situation. For this reason, in our formulation, we have replaced the measure σ by a general measure μ. For more details, we refer to [1,3,4].

In this article, assuming that α and β in (1.1) are also Young functions (not necessarily $\mathcal{N}$-functions) in the sense of [5], and denoting by λ the N-dimensional Lebesgue measure, we show that a necessary condition for the existence of weak solutions of (1.1) is that

$$\int_\Omega F_1 \mathrm{d}x + \int_{\partial\Omega} F_2 \mathrm{d}\mu \in \lambda(\Omega)R(\alpha) + \mu(\partial\Omega)R(\beta) \tag{1.2}$$

while a sufficient condition is that

$$\int_\Omega F_1 \mathrm{d}x + \int_{\partial\Omega} F_2 \mathrm{d}\mu \in \mathrm{Int}\{\lambda(\Omega)R(\alpha) + \mu(\partial\Omega)R(\beta)\}, \tag{1.3}$$

where in both cases the given functions $F_1 \in L^{q_1}(\Omega)$, $F_2 \in L^{q_2}(\partial\Omega, \mathrm{d}\mu)$ with

$$\begin{cases} q_1 = q_2 = 1 & \text{if } p \in (N, \infty), \\ q_1, q_2 \in (1, \infty) & \text{if } p = N, \\ q_1 = Np/(N(p-1)+p),\ \ q_2 = dp/(p(d+1)-N) & \text{if } p \in (1, N). \end{cases}$$

The conditions (1.2) and (1.3) are an extension of the classical Landesman–Lazer [6] condition for the solvability of the classical Neumann problem with zero boundary data, that is, $p = 2$ and $\mu \equiv 0$. This result has been extended by Brezis and Haraux [7] where they have established an elegant abstract setting by using the notion of the subdifferential of proper convex lower-semicontinuous functionals on Hilbert spaces. Here, since the given functions are not necessary in a Hilbert space, we will need to describe and establish some properties of the subdifferential of proper convex lower-semicontinuous functionals on Banach spaces.

We organize the paper as follows. In Section 2, we give some basic definitions and establish some preliminary results on Sobolev spaces and the subdifferential of proper convex lower-semicontinuous functionals on Banach spaces as they are needed throughout the paper. In Section 3, we state and prove the main results.

2. Preliminaries

2.1. Generalized normal derivatives

In this article, we allow rather bad boundaries, such as for example the boundary of the von Koch snowflake. Hence, it is not always possible to define an outer normal vector or a weak normal derivative even for smooth functions such as functions in $C^\infty(\overline{\Omega})$. So we need to replace the normal derivative with the *p-generalized normal derivative* $\mathsf{N}_p(u)$, which is given for *smooth domains* and *smooth functions* by

$$\mathrm{d}\mathsf{N}_p(u) = |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} \mathrm{d}\sigma.$$

Here σ denotes the *surface measure* and ν the *outer normal vector* to Ω. The p-generalized normal derivative has been first introduced in [3].