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Coincidence criteria of the maximal stable bridges in the approach problems☆

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ABSTRACT

Two person antagonistic differential games with dynamics described by nonlinear ordinary differential equations are considered. For a given compact target set $M \subset \mathbb{R}^n$ the problem of approach at a given time instant θ and the problem of approach to a given time instant θ are studied. For these problems, the coincidence criteria of maximal u-stable bridges are obtained.

1. Introduction

The conflict-control system, whose behavior is described by an ordinary differential equation, is considered. Two game-approach problems with given terminal target set $M \subset \mathbb{R}^n$ are studied and compared. The first approach problem posed for the first player consists of choosing a positional strategy such that every motion generated by this strategy hits the target set M at the instant of time θ. The second approach problem posed for the first player consists of choosing a positional strategy such that every motion generated by this strategy hits the target set M no later than the instant of time θ (see, [1,2]). The second player attempts to prevent the first player. Briefly, the first approach problem is called the approach problem at the instant of time θ, the second one is called the approach problem to the instant of time θ. Note that these problems are fundamental ones in differential game theory and are related to many important problems of guaranteed control when the system is subjected to perturbations (see, [1–3]). Moreover, it is possible to formulate many of the concrete differential games in the framework of these problems (see, [4]).

The first problem is simpler than the second one and consequently it is comparatively easier to obtain the solution methods of the first problem than the second one. Thus, it is natural to derive the conditions concerning the conflict-control system and target set M, guaranteeing the coincidence of the solutions of these approach problems. Under such conditions it is reasonable to develop methods giving the solution of the first problem.

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The basic element of the method suggested in [1,2] and giving a solution of the approach problems, is maximal u-stable bridge, i.e. the positional absorption set notion. If the appropriate maximal u-stable bridge is built, then it is relatively simple to define the positional strategy which solves the given approach problem. It is known that (see, [1,2]) in this situation, the positional strategy solving the given approach problem is constructed as extremal strategy to maximal u-stable bridge. In this study, the conditions which guarantee the coincidence of the maximal u-stable bridges in approach problems for autonomous conflict-control systems, are obtained. The grounds of these criteria are based on the constructions and results developed in [5–20]. For example, in the definitions of the stable absorption operator and u-stable bridge, unification constructions developed in [10,11,15,16,20] are used. Throughout the paper the infinitesimal form definition of u-stable bridges, introduced in [8,9], is also used. The subject of the investigation of this article is close to the studies carried out in [1–31]. The paper is organized as follows:

In Section 2, using the stable absorption operator notion, definitions of the u-stable bridges in M-approach problem at the instant of time θ and in M-approach problem to the instant of time θ are given. These definitions are formulated in "forward" and "backward" time terms (Definitions 2.2, 2.4, 2.6 and 2.8).

In Section 3, contingent cone and derivative sets concepts which are used in the set valued analysis, are introduced and some interconnections between these notions are studied (Propositions 3.2 and 3.3).

In Section 4, it is proved that the sum of contingent cone of the attainable set and the right-hand side of the given differential inclusion equals the right-hand side derivative set of the set valued map, the graph of which is an integral funnel of the differential inclusion (Theorem 4.1). Necessary conditions for u-stability of a given set are obtained. These conditions have infinitesimal construction and connect the derivative set and contingent cone of the given set valued map and the right-hand side of the conflict-control system (Theorems 4.2 and 4.3).

In Section 5 the conditions guaranteeing the coincidence of the maximal u-stable bridges in the M-approach problems at the instant of time θ and to the instant of time θ for autonomous conflict-control systems are obtained (Theorems 5.1–5.3). In Theorem 5.1 it is proved that the maximal u-stable bridges in the M-approach problems at the instant of time θ and to the instant of time θ for autonomous conflict-control system coincide, iff the set $[t_0, \theta] \times M$ is u-stable with respect to the conflict-control system (i.e. the set M is weakly invariant with respect to the family of differential inclusions, defined by given conflict-control system). Theorem 5.2 asserts that the maximal u-stable bridges in the M-approach problems at the instant of time θ and to the instant of time θ for the autonomous conflict-control system coincide, iff the Hamiltonian of the system is not negative for every $x \in M$. In Theorem 5.3 it is shown that the maximal u-stable bridges in the M-approach problems at the instant of time θ and to the instant of time θ for autonomous conflict-control system coincide, iff the maximal u-stable bridge in the M-approach problem at the instant of time θ defines a monotone set valued map.

In Section 6, when the target set M has a piecewise-smooth boundary, the coincidence condition of the maximal u-stable bridges in the M-approach problems at the instant of time θ and to the instant of time θ is given (Theorem 6.1).

For $x_0 \in \mathbb{R}^n$, $r > 0$ we denote

$$B_n(x_0, r) = \left\{x \in \mathbb{R}^n : \|x - x_0\| \le r\right\},$$
$$B_n = \left\{x \in \mathbb{R}^n : \|x\| \le 1\right\}, \quad S_n = \left\{x \in \mathbb{R}^n : \|x\| = 1\right\}$$

where $\|\cdot\|$ denotes the Euclidean norm.

By $2^{\mathbb{R}^n}$ we denote the family of all subsets of the space $\mathbb{R}^n$. The family of nonempty compact convex subsets of $\mathbb{R}^n$ is denoted by $\text{conv}(\mathbb{R}^n)$.

The Hausdorff distance between the sets $A \subset \mathbb{R}^n$ and $E \subset \mathbb{R}^n$ is denoted by $h(A, E)$ and is defined as

$$h(A, E) = \max\left\{\sup_{x \in A} \text{dist}(x, E), \sup_{y \in E} \text{dist}(y, A)\right\}$$

where $\text{dist}(x, E) = \inf\{\|x - y\| : y \in E\}$.

For given $E \subset \mathbb{R}^n$ and $l \in \mathbb{R}^n$ we set

$$\sigma(E, l) = \sup_{x \in E} \langle x, l\rangle \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ denotes scalar product.

Consider conflict-control system the behavior of which is described by a differential equation

$$\dot{x} = f(t, x, u, v) \tag{1.2}$$

where $x \in \mathbb{R}^n$ is the phase state vector of the system, $u \in P$ is the control vector of the first player, $v \in Q$ is the control vector of the second player, $P \subset \mathbb{R}^p$, $Q \subset \mathbb{R}^q$ are compact sets, $t \in [t_0, \theta]$ is the time. The pair $(t, x) \in [t_0, \theta] \times \mathbb{R}^n$ is called a position of the system. It is assumed that the right-hand side of the system (1.2) satisfies the following conditions:

1.A. The function $f(\cdot) : [t_0, \theta] \times \mathbb{R}^n \times P \times Q \to \mathbb{R}^n$ is continuous and for every compact set $D \subset [t_0, \theta] \times \mathbb{R}^n$ there exists $L = L(D) > 0$ such that

$$\|f(t, x_1, u, v) - f(t, x_2, u, v)\| \le L\,\|x_1 - x_2\|$$

for any $(t, x_1, u, v) \in D \times P \times Q$, $(t, x_2, u, v) \in D \times P \times Q$;