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Equivalent conditions for generalized contractions on (ordered) metric spaces

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ABSTRACT

We establish a geometric lemma giving a list of equivalent conditions for some subsets of the plane. As its application, we get that various contractive conditions using the so-called altering distance functions coincide with classical ones. We consider several classes of mappings both on metric spaces and ordered metric spaces. In particular, we show that unexpectedly, some very recent fixed point theorems for generalized contractions on ordered metric spaces obtained by Harjani and Sadarangani [J. Harjani, K. Sadarangani, Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations, Nonlinear Anal. 72 (2010) 1188–1197], and Amini-Harandi and Emami [A. Amini-Harandi, H. Emami A fixed point theorem for contraction type maps in partially ordered metric spaces and application to ordinary differential equations, Nonlinear Anal. 72 (2010) 2238–2242] do follow from an earlier result of O'Regan and Petruşel [D. O'Regan and A. Petruşel, Fixed point theorems for generalized contractions in ordered metric spaces, J. Math. Anal. Appl. 341 (2008) 1241–1252].

1. Introduction

Let (X, d) be a complete metric space and T be a selfmap of X. Following Rus [1] T is called a *Picard operator* if T has a unique fixed point x_* and for any $x \in X$, $\lim_{n\to\infty} T^n x = x_*$. Given a function φ from $\mathbb{R}_+$, the set of all nonnegative reals, into $\mathbb{R}_+$ such that $\varphi(t) < t$ for all $t > 0$, we say that T is a *φ-contraction* if

$$d(Tx, Ty) \le \varphi(d(x, y)) \quad \text{for any } x, y \in X. \tag{1}$$

The following generalization of the Banach fixed point theorem was obtained in 1968 by Browder [2]: if φ is right continuous and nondecreasing, then any φ-contraction is a Picard operator. Subsequently, this result was extended in 1969 by Boyd and Wong [3], who observed that it sufficed to assume only the right-upper semicontinuity of φ. Independently, the following contractive condition was introduced by Krasnoselskiĭ et al. [4]:

$$d(Tx, Ty) \le d(x, y) - \eta(d(x, y)) \quad \text{for any } x, y \in X, \tag{2}$$

where $\eta: \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function such that $\eta^{-1}(\{0\}) = \{0\}$. Clearly, this is a special form of the Boyd–Wong condition with $\varphi(t) := t - \eta(t)$ for $t \in \mathbb{R}_+$. Next, condition (2) was rediscovered in 2001 by Rhoades [5], who assumed additionally that η is nondecreasing. (See also [6]; here η is continuous and nondecreasing, and such that $\lim_{t\to\infty} \eta(t) = \infty$.)

On the other hand, in 1976 Delbosco [7] initiated a study of the following contractive condition with the so-called altering distance function ψ:

$$\psi(d(Tx, Ty)) \le \alpha\psi(d(x, y)) \quad \text{for any } x, y \in X \text{ and some } \alpha \in [0, 1), \tag{3}$$

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where $\psi \in \mathfrak{F}$ and $\mathfrak{F}$ is defined by

$$\mathfrak{F} := \{\psi: \mathbb{R}_+ \to \mathbb{R}_+ : \ \psi \text{ is continuous, nondecreasing and } \psi^{-1}(\{0\}) = \{0\}\}. \tag{4}$$

In fact, Delbosco [7] considered only the case in which ψ is a power function. Subsequently, his result was slightly extended by Skof [8] in 1977. At last, the above family $\mathfrak{F}$ was defined in 1984 by Khan et al. [9], who proved that any mapping T satisfying (3) with $\psi \in \mathfrak{F}$ is a Picard operator. (Actually, they used a more general contractive condition.)

Recently, Dutta and Choudhury [10] obtained the following fixed point theorem with the intention to subsume the results of Rhoades [5] and Khan et al. [9].

Theorem 1 ([10]). *Let T be a selfmap of a complete metric space (X, d) such that*

$$\psi(d(Tx, Ty)) \le \psi(d(x, y)) - \eta(d(x, y)) \quad \text{for any } x, y \in X, \tag{5}$$

where $\psi, \eta \in \mathfrak{F}$. Then, T is a Picard operator.

However, in this paper we show that unexpectedly, Theorem 1 is in fact identical with the theorem of Khan et al. [9]. Moreover, both these results turn out to be equivalent to Browder's [2] theorem, which means that the three theorems deal with the same class of mappings. In particular, if T satisfies (3) or (5), then there exists a continuous and nondecreasing function $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$ such that T is a φ-contraction, i.e., (1) holds. Here we use a similar approach as in [11] and our papers [12–15]. First, we establish a geometric lemma giving a list of equivalent conditions for some subsets of the quadrant $\mathbb{R}_+^2$. Then, applying the lemma to the set

$$\{(d(x, y), d(Tx, Ty)) : \ x, y \in X\},$$

we get a result on the equivalence between several contractive conditions for a mapping T. Next, by substituting $d(Tx, Sy)$ for $d(Tx, Ty)$, and $M(x, y)$ for $d(x, y)$ in the above set, where

$$M(x, y) := \max\{d(x, y), d(x, Tx), d(y, Sy), (d(y, Tx) + d(x, Sy))/2\},$$

we obtain with the help of the lemma a list of equivalent contractive conditions for two mappings S and T, from which we may infer that a very recent generalization of the Dutta–Choudhury theorem due to Dorić [16] is identical with earlier results of Zhang [17] and Zhang and Song [18]. Moreover, each of these theorem is a particular case of our [19, Theorem 3.8].

On the other hand, applying the lemma to the set

$$\{(d(x, y), d(Tx, Ty)) : \ x, y \in X, \ x \preceq y\},$$

where $\preceq$ is a partial order in X, we get the result, which illuminates relations between several recent fixed point theorems for mappings on ordered metric spaces. The first such a theorem was obtained by Ran and Reurings [20] in 2004. Subsequently, their result was generalized by a number of authors (see [21–32]). Here we recall the following fixed point theorem of O'Regan and Petruşel [31, Theorem 3.6], which is a counterpart of Matkowski's [33, Theorem 1.2] (see also [34, p. 15]) for mappings on ordered metric spaces.

Theorem 2. *Let $(X, d, \preceq)$ be an ordered complete metric space and $T: X \to X$ be increasing (with respect to $\preceq$) and such that $x_0 \preceq Tx_0$ for some $x_0 \in X$, and*

$$d(Tx, Ty) \le \varphi(d(x, y)) \quad \text{for any } x, y \in X \text{ with } x \preceq y,$$

where a function $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing and such that $\lim_{n\to\infty} \varphi^n(t) = 0$ for any $t > 0$. Assume that for any $x, y \in X$, there exists a lower bound or an upper bound for $\{x, y\}$. If T is continuous or for any increasing sequence (x_n), $x_n \to x$ implies $x_n \preceq x$ for all $n \in \mathbb{N}$, then T is a Picard operator.

With the help of our geometric lemma, we shall show that unexpectedly, Theorem 2 subsumes very recent results of Harjani and Sadarangani ([24, Theorem 4] and [25, Theorem 2.3]), and Amini-Harandi and Emami [22, Theorem 2.1], probably, contrary to the expectations of the authors, who were familiar with the paper [31]. Moreover, we shall also present that [24, Theorem 4] does yield a subsequent [22, Theorem 2.1].

2. Equivalent conditions for a subset of the plane

Given functions $\psi, \eta: \mathbb{R}_+ \to \mathbb{R}_+$, we set

$$E_{\psi,\eta} := \{(t, u) \in \mathbb{R}_+^2 : \ \psi(u) \le \psi(t) - \eta(t)\}$$

and

$$E_{\psi} := \{(t, u) \in \mathbb{R}_+^2 : \ u \le \psi(t)\}.$$

Let the family $\mathfrak{F}$ be defined by (4). The following geometric lemma for a subset of the plane is our crucial result.