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Reversibility and quasi-homogeneous normal forms of vector fields

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ABSTRACT

This paper uses tools in quasi-homogeneous normal form theory to discuss certain aspects of reversible vector fields around an equilibrium point. Our main result provides an algorithm, via Lie Triangle, that detects the non-reversibility of vector fields. That is, it is possible to decide whether a planar center is not reversible. Some of the theory developed is also applied to get further results on nilpotent and degenerate polynomial vector fields. We find several families of nilpotent centers which are non-reversible.

1. Introduction and setting of the problem

This paper is focused on the differential systems with time-reversal symmetries. A time-reversal symmetry is one of the fundamental symmetries in natural science and it arises in many branches of physics; see for instance, [1] for a survey on reversible systems and related topics.

In the last decades there has been an increasing interest in the study of systems with time-reversal symmetries. In recent years, a lot of attention has been devoted to understand and use the interplay between dynamic and symmetry properties. Reversible vector fields were first considered by Birkhoff, in the beginning of the last century, when he was studying the restricted three body problem. In [2] the theory was formalized by Devaney.

The property of reversibility of a planar vector field is a sufficient condition so that a monodromic planar vector field to be a center, and this provides a strong motivation to study the reversibility of a vector field, since the center problem is one of the main open problems in the qualitative theory of planar dynamical systems. Moreover, there exists a strong connection between the reversibility and the center problem of a planar vector field. In fact, it is known that a planar system having a non-degenerate (respectively nilpotent) center at the origin is reversible (respectively orbitally reversible); see [3,4]. Therefore, the reversibility problem can provide new information in the center problem. In this paper, we also study nilpotent centers which are non-reversible.

Much effort has been dedicated to understand the connection between centers, analytic integrability and reversibility of a planar vector field; see for instance ([5,4,6–11], and the references therein).

On the other hand, much work has been done in the study of polynomial vector fields by means of techniques in the quasi-homogeneous normal form theory; see for instance, [12,5,13–15].

In this paper, our main aim is to establish a discussion involving reversible vector fields and quasi-homogeneous normal form theory.

Now we need to introduce some definitions and terminology.

- An involution is a diffeomorphism $\sigma \in \mathcal{C}^\infty(U_0 \subset \mathbb{R}^n, \mathbb{R}^n)$, such that $\sigma \circ \sigma = Id$, where U_0 is a neighborhood of $0 \in \mathbb{R}^n$. Denote $\mathrm{Fix}\,(\sigma) = \{\mathbf{x} \in U_0 | \sigma(\mathbf{x}) = \mathbf{x}\}$. This set is a local sub-manifold of $\mathbb{R}^n$ and we are assuming throughout the paper that $\dim(\mathrm{Fix}\,(\sigma)) = n - 1$.

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- We say that the system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$, or the vector field $\mathbf{F}$ is reversible if there is an involution σ, $\sigma(\mathbf{0}) = \mathbf{0}$, such that $\sigma_* \mathbf{F} = -\mathbf{F}$.
- We say that the system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$, or the vector field $\mathbf{F}$ is orbitally reversible if there exists an involution σ and a function $f \in \mathcal{C}^\infty(U_0 \subset \mathbb{R}^n, \mathbb{R})$, $f(\mathbf{0}) = 1$ such that $\sigma_*(f\mathbf{F}) = -f\mathbf{F}$.
- We say that the system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$, or the vector field $\mathbf{F}$ is reversible with respect to the coordinate x_i (or just R_{x_i}-reversible), $i = 1, \ldots, n$, if it is reversible with respect to the involution

$$R_{x_i}(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) = (x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n).$$

We mean that the system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ is invariant under a coordinate system given by $x_i \to -x_i$, $t \to -t$ for some i.

We deal with n-dimensional systems. Let $\mathbf{F}_0 = (X, Y)$, $X \in \mathbb{R}$, $Y \in \mathbb{R}^{n-1}$, a (germ of) C^r reversible vector field with $\mathbf{F}_0(\mathbf{0}) = \mathbf{0}$, $r > 1$, $r = \infty$ or $r = \omega$. We know (Montgomery–Bochner Theorem, (see [16], pp. 206)) that there exists a coordinate system of class C^r around 0 such that the vector field is expressed as $\mathbf{F}_0(x, y) = (f(x^2, y), xg(x^2, y))$, $x \in \mathbb{R}$, $y \in \mathbb{R}^{n-1}$ with f and g being C^r-functions. So a system is not reversible provided that it cannot be transformed, up to C^r-*conjugacy*, to the above form. This is, roughly speaking, the route we have chosen to conduct this paper.

In summary, in what follows, we give a rough all-over description of the main results of the paper.

- *Necessary conditions of reversibility*. We prove that in order to calculate necessary conditions for the reversibility of a vector is enough to use reversible generators. (Theorem 2.11). This fact provides a strong simplification in the computation of reversible vector fields.
- *Algorithm of non-reversibility*. We exhibit an algorithm, via the Lie triangle, that detects the non-reversibility of the system (Theorem 3.19).
- *Applications*. We apply some of the theory developed to get further results on nilpotent and degenerate planar and tridimensional vector fields.

The remaining sections are organized as follows. In Section 2, some terminology, basic concepts and preparatory results are presented. In Section 3, an adequate normal form to detect the reversibility of a vector field is discussed. In Section 4, we present some applications on nilpotent and degenerate planar vector fields, in this case the center and the reversibility problem are connected. We also analyze the reversibility problem for a nilpotent family of tridimensional vector fields.

2. *N*-reversibility

First of all, we establish some terminology and definitions.

Let $\mathscr{P}_k^{\mathbf{t}}$ be the vector space of real quasi-homogeneous polynomial functions of degree $k \in \mathbb{N}$, respect to the type $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{N}^n$, i.e., $f \in \mathscr{P}_k^{\mathbf{t}}$, if and only if $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is either $f \equiv 0$ or f is a polynomial verifying $f(\epsilon^{t_1}x_1, \ldots, \epsilon^{t_n}x_n) = \epsilon^k f(x_1, \ldots, x_n)$ for all $\epsilon, x_1, \ldots, x_n \in \mathbb{R}$, and $\mathscr{Q}_k^{\mathbf{t}}$ be the vector space of the polynomial quasi-homogeneous vector fields of degree $k \in \mathbb{Z}$, respect to type $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{N}^n$, i.e., $\mathbf{F} = (Q_1, \ldots, Q_n)^T \in \mathscr{Q}_k^{\mathbf{t}}$, $\mathbf{F} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, if and only if $Q_i \in \mathscr{P}_{k+t_i}^{\mathbf{t}}$, $\forall i = 1, \ldots, n$. For more details see [17].

We will denote:

- $\mathscr{O}_k^{\mathbf{t}} := \left\{\mu_k \in \mathscr{P}_k^{\mathbf{t}} \,|\, \mu_k(-x, y) = -\mu_k(x, y)\right\}$.
- $\mathscr{E}_k^{\mathbf{t}} := \left\{\mu_k \in \mathscr{P}_k^{\mathbf{t}} \,|\, \mu_k(-x, y) = \mu_k(x, y)\right\}$, where $x = x_1$ and $y = (x_2, \ldots, x_n)^T$.
- $\mathscr{R}_k^{\mathbf{t}} := \left\{(p, q)^T \in \mathscr{Q}_k^{\mathbf{t}} | p \in \mathscr{E}_{k+t_1}^{\mathbf{t}}, q_i \in \mathscr{O}_{k+t_i}^{\mathbf{t}}, i = 2, \ldots, n\right\}$, the R_x-reversible quasi-homogeneous vector fields of degree k, where $q = (q_2, \ldots, q_n)^T$.
- $\mathscr{S}_k^{\mathbf{t}} := \left\{(p, q)^T \in \mathscr{Q}_k^{\mathbf{t}} : p \in \mathscr{O}_{k+t_1}^{\mathbf{t}}, q_i \in \mathscr{E}_{k+t_i}^{\mathbf{t}}, i = 2, \ldots, n\right\}$ the R_x-symmetric quasi-homogeneous vector fields of degree k.

In this way we may always consider the decomposition $\mathscr{P}_k^{\mathbf{t}} = \mathscr{O}_k^{\mathbf{t}} \bigoplus \mathscr{E}_k^{\mathbf{t}}$ and $\mathscr{Q}_k^{\mathbf{t}} = \mathscr{R}_k^{\mathbf{t}} \bigoplus \mathscr{S}_k^{\mathbf{t}}$.

Remark 2.1. Denote $\widetilde{\mathbf{U}}_j = \mathrm{Proy}_{\mathscr{R}_j^{\mathbf{t}}}(\mathbf{U})$ and $\overline{\mathbf{U}}_j = \mathrm{Proy}_{\mathscr{S}_j^{\mathbf{t}}}(\mathbf{U})$.

Next lemma is a direct consequence of the last definitions.

Lemma 2.2. *Let* $S = \mathrm{diag}(-1, \overbrace{1, \ldots, 1}^{n-1})$. *Then*
(a) $\tilde{\mu}_k \in \mathscr{O}_k^{\mathbf{t}}$ *if and only if* $\tilde{\mu}_k(S\mathbf{x}) = -\tilde{\mu}_k(\mathbf{x})$.
(b) $\overline{\mu}_k \in \mathscr{E}_k^{\mathbf{t}}$ *if and only if* $\overline{\mu}_k(S\mathbf{x}) = \overline{\mu}_k(\mathbf{x})$.
(c) $\widetilde{\mathbf{P}}_k \in \mathscr{R}_k^{\mathbf{t}}$, *if and only if* $\widetilde{\mathbf{P}}_k(S\mathbf{x}) = -S\widetilde{\mathbf{P}}_k(\mathbf{x})$.
(d) $\overline{\mathbf{P}}_k \in \mathscr{S}_k^{\mathbf{t}}$, *if and only if* $\overline{\mathbf{P}}_k(S\mathbf{x}) = S\overline{\mathbf{P}}_k(\mathbf{x})$.

Lemma 2.3. *Let* $\widetilde{\mathbf{F}}_r \in \mathscr{R}_r^{\mathbf{t}}$, $\overline{\mathbf{F}}_s \in \mathscr{S}_s^{\mathbf{t}}$, $\tilde{\mu}_k \in \mathscr{O}_k^{\mathbf{t}}$, $\overline{\mu}_k \in \mathscr{E}_k^{\mathbf{t}}$. *Hence:*
(a) $\nabla\tilde{\mu}_k \cdot \widetilde{\mathbf{F}}_r \in \mathscr{E}_{r+k}^{\mathbf{t}}$.
(b) $\nabla\tilde{\mu}_k \cdot \overline{\mathbf{F}}_s \in \mathscr{O}_{s+k}^{\mathbf{t}}$.