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Interpolation of bilinear operators and compactness

Dicesar Lass Fernandez [a,1], Eduardo Brandani da Silva [b,*]

[a] IMECC-UNICAMP, Caixa Postal 6065, Campinas - SP, 13083-859, Brazil

[b] DMA-UEM, Av. Colombo 5790, Maringa-PR, 87030-110, Brazil

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ABSTRACT

The behavior of bilinear operators acting on interpolation of Banach spaces for the ρ method in relation to the compactness is analyzed. Similar results of Lions-Peetre, Hayakawa and Persson's compactness theorems are obtained for the bilinear case and the ρ method.

1. Introduction

The study of compactness of multilinear operators for interpolation spaces goes back to Calderón [1, pp. 119–120]. Under an approximation hypothesis, Calderón established a one-side type general result, but restricted to complex interpolation spaces.

For the real method, if $\mathbf{E} = (E_0, E_1)$, $\mathbf{F} = (F_0, F_1)$ and $\mathbf{G} = (G_0, G_1)$ are Banach couples, a classical result by Lions–Peetre assures that if T is a bounded bilinear operator from $(E_0 + E_1) \times (F_0 + F_1)$ into $G_0 + G_1$, whose restrictions $T|E_k \times F_k$ $(k = 0, 1)$ are also bounded from $E_k \times F_k$ into G_k $(k = 0, 1)$, then T is bounded from $\mathbf{E}_{\theta,p;J} \times \mathbf{F}_{\theta,q;J}$ into $\mathbf{G}_{\theta,r;J}$, where $0 < \theta < 1$ and $1/r = 1/p + 1/q - 1$. Lately several authors have obtained new and more general results for interpolation of bilinear and multilinear operators, for example see [2–4].

On the other hand, the behaviour of compact multilinear operators under real interpolation functors or more general functors does not seem to have been investigated yet. This is our main subject in this work.

After some preliminaries on interpolation of linear and bilinear operators, generalizations of Lions–Peetre compactness theorems [5, Theorem V.2.1] (the one with the same departure spaces) and [5, Theorem V.2.2] (the one with the same arriving spaces) will be stated. The proof of the first one is an adaptation of the original proof, but the latter requires an involved argument.

Thereafter, a two-side result for general interpolation functors of type ρ, with the additional cost of an approximation hypothesis on the departure Banach couples, will then be given. Thus, a theorem of Hayakawa type (i.e. a two-side result without approximation hypothesis) will be obtained. The point at this issue is that the Hayakawa type theorem is nothing but a corollary of the result with approximation hypothesis. Consequently, a one-side result holds for ordered Banach couples.

* Corresponding author. Tel.: +55 44 32614681.
E-mail addresses: dicesar@ime.unicamp.br (D.L. Fernandez), ebsilva@wnet.com.br (E.B. da Silva).
[1] Tel.: +55 19 35216051.

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Finally, as a consequence of the second lemma of Lions–Peetre type, a compactness theorem of Persson type is obtained. To avoid ponderous notations we have restricted ourselves to the bilinear case. A generalization for the ρ method of the Lions–Peetre's bilinear theorem will also be provided in this work.

2. Preliminaries on interpolation

2.1. Interpolation functors

A pair of Banach spaces $\mathbf{E} = (E_0, E_1)$ is said to be a Banach couple if E_0 and E_1 are continuously embedded in some Hausdorff linear topological space $\mathcal{E}$. Then we can form their intersection $E_0 \cap E_1$ and sum $E_0 + E_1$; it can be seen that $E_0 \cap E_1$ and $E_0 + E_1$ become Banach spaces when endowed with the norms

$$\|x\|_{E_0\cap E_1} = \max\{\|x\|_{E_0}, \|x\|_{E_1}\},$$

and

$$\|x\|_{E_0+E_1} = \inf_{x=x_0+x_1}\{\|x_0\|_{E_0} + \|x_1\|_{E_1}\},$$

respectively.

We shall say that a Banach space is an *intermediate space* with respect to a Banach couple $\mathbf{E} = (E_0, E_1)$ if

$$E_0 \cap E_1 \hookrightarrow E \hookrightarrow E_0 + E_1.$$

(The hookarrow $\hookrightarrow$ denotes bounded embeddings).

Let $\mathbf{E} = (E_0, E_1)$ and $\mathbf{F} = (F_0, F_1)$ be Banach couples. We shall denote by $L(\mathbf{E}, \mathbf{F})$ the set of all linear mappings from E_0+E_1 into $F_0 + F_1$ such that $T|_{E_K}$ is *bounded* from E_k into F_k, $k = 0, 1$.

By an *interpolation functor* $\mathcal{F}$ we shall mean a functor which to each Banach couple $\mathbf{E} = (E_0, E_1)$ associates an intermediate space $\mathcal{F}(E_0, E_1)$ between E_0 and E_1, and such that $T|_{\mathcal{F}(E_0,E_1)} \in L(\mathcal{F}(E_0, E_1), \mathcal{F}(F_0, F_1))$, for all $T \in L(\mathbf{E}, \mathbf{F})$.

The interpolation functors which we shall consider depend on function parameters. For more details about interpolation see [6].

2.2. The function parameters

By a *function parameter* ρ we shall mean a continuous and positive function on $\mathbb{R}_+$.

We shall say that a function parameter ρ belongs to the class $\mathcal{B}$, if it satisfies the following conditions:

$$\rho(1) = 1, \tag{1}$$

and

$$\overline{\rho}(s) = \sup_{t>0} \frac{\rho(st)}{\rho(t)} < +\infty, \quad s > 0. \tag{2}$$

Also, we shall say that a function parameter $\rho \in \mathcal{B}$ belongs to the class $\mathcal{B}^{+-}$ if it satisfies

$$\int_0^\infty \min\left(1, \frac{1}{t}\right) \overline{\rho}(t) \frac{dt}{t} < +\infty. \tag{3}$$

From (1)–(3) we see that $\mathcal{B}^{+-}$ is contained in Peetre's class $\mathcal{P}^{+-}$, i.e. the class of pseudo-concave function parameters which satisfies

$$\overline{\rho}(t) = o(\max(1, t)). \tag{4}$$

(See [7,8].)

The function parameter $\rho_\theta(t) = t^\theta$, $0 \le \theta \le 1$, belongs to $\mathcal{B}$. It corresponds to the usual parameter θ. Further, $\rho_\theta \in \mathcal{B}^{+-}$ if $0 < \theta < 1$, but $\rho_0, \rho_1 \notin \mathcal{B}^{+-}$.

To control function parameters we shall need to recall the Boyd indices (see [9,10,3]).

2.3. The Boyd indices

Given a function parameter $\rho \in \mathcal{B}$, the *Boyd indices* $\alpha_{\overline{\rho}}$ and $\beta_{\overline{\rho}}$ of the submultiplicative function $\overline{\rho}$ are defined, respectively, by

$$\alpha_{\overline{\rho}} = \sup_{1<t<\infty} \frac{\log \overline{\rho}(t)}{\log t}, \tag{5}$$

and

$$\beta_{\overline{\rho}} = \sup_{0<t<1} \frac{\log \overline{\rho}(t)}{\log t}. \tag{6}$$