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Orbital stability of peakons for the Degasperis–Procesi equation with strong dispersion☆

Yanggeng Fu *, Zhengrong Liu, Xingyu Yang

Department of Mathematics, South China University of Technology, Guangzhou 510640, PR China

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ABSTRACT

In this paper, we study the orbital stability of the peakons for the Degasperis–Procesi equation with a strong dispersive term on the line. Using the method in [Z. Lin, Y. Liu, Stability of peakons for the Degasperis–Procesi equation, Comm. Pure Appl. Math. 62 (2009) 125–146], we prove that the shapes of these peakons are stable under small perturbations. Some previous results are extended.

1. Introduction

In [1], Degasperis and Procesi studied the family of third-order dispersive partial differential equation (PDE) conservation laws

$$u_t - \alpha^2 u_{xxt} + \gamma_0 u_{xxx} + c_0 u_x = (c_1 u^2 + c_2 u_x^2 + c_3 u u_{xx})_x. \tag{1.1}$$

Within this family, only three equations that satisfy asymptotic integrability conditions up to third order were singled out. After rescaling and applying a Galilean transformation, these equations are the Korteweg–de Vries (KdV) equation,

$$u_t + u_{xxx} + u u_x = 0,$$

the Camassa–Holm (CH) shallow water equation,

$$u_t - u_{txx} + 3u u_x = 2u_x u_{xx} + u u_{xxx} \tag{1.2}$$

and the Degasperis–Procesi (DP) equation,

$$u_t - u_{txx} + 4u u_x = 3u_x u_{xx} + u u_{xxx}, \quad t > 0,\ x \in \mathbb{R}. \tag{1.3}$$

These three cases are all the completely integrable candidates for (1.1) by Painlevé analysis. By constructing a Lax pair and a bi-Hamiltonian structure, Degasperis et al. [2] showed the formal integrability of the DP equation as Hamiltonian systems.

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* Corresponding author.
E-mail address: fu.yg@mail.scut.edu.cn (Y. Fu).

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The CH equation was first derived by Fokas and Fuchssteiner [3] as a bi-Hamiltonian system, and then by Camassa and Holm [4] as a model for shallow water waves. The DP equation is also an approximation to the incompressible Euler equations for shallow water [5–8] in dimensionless space–time variables (x, t) and its asymptotic accuracy is the same as that of the CH shallow water equation, where both solutions $u(t, x)$ for the DP equation and the CH equation are considered as the horizontal component of the fluid velocity at time t in the spatial x-direction with momentum density, but evaluated at the different level lines of the fluid domain [8].

As is well known, the KdV equation is an integrable Hamiltonian equation that possesses smooth solitons as traveling waves. In the KdV equation, the leading-order asymptotic balance that confines the traveling wave solitons occurs between nonlinear steepening and linear dispersion. However, the nonlinear dispersion and nonlocal balance in the CH equation and the DP equation, even in the absence of linear dispersion, can still produce confined solitary traveling waves

$$u_1(t, x) = c\mathrm{e}^{-|x-ct|}, \tag{1.4}$$

where c is the constant wave speed. Because of their shapes (they are smooth except for a peak at their crest), these solutions are called peakons [2,4]. The peakons of both equations are true solitons that interact via elastic collisions under the CH dynamics or the DP dynamics, respectively.

It is worthwhile noting that the DP equation has not only peakons, but also shock peakons [9] of the form

$$u(t, x) = -\frac{1}{t+k}\operatorname{sgn}(x)\mathrm{e}^{-|x|}, \quad k > 0.$$

This feature is regarded as a significant difference between the DP equation and the CH equation.

The stability of solitary waves is one of the fundamental properties of the solutions of nonlinear wave equations. Because a small perturbation of a solitary wave can yield another one with a different speed and phase shift, the appropriate notion of stability is orbital stability. That is, a wave starting close to a solitary wave remains close to some translate of it at all later times. Thus the shape of the wave remains approximately the same for all times. For the CH equation and the DP equation, their conservation laws play an important role in the study of the stability of peakons. The following are three useful conservation laws of the DP equation:

$$E_1(u) = \int_{\mathbb{R}} y\mathrm{d}x, \qquad E_2(u) = \int_{\mathbb{R}} yv\mathrm{d}x, \qquad E_3(u) = \int_{\mathbb{R}} u^3\mathrm{d}x, \tag{1.5}$$

where $y = (1 - \partial_x^2)u$ and $v = (4 - \partial_x^2)^{-1}u$, while the corresponding three useful conservation laws of the CH equation are the following:

$$F_1(u) = \int_{\mathbb{R}} y\mathrm{d}x, \qquad F_2(u) = \int_{\mathbb{R}} (u^2 + u_x^2)\mathrm{d}x, \qquad F_3(u) = \int_{\mathbb{R}} (u^3 + uu_x^2)\mathrm{d}x. \tag{1.6}$$

Using the conservation laws F_2 and F_3, Constantin and Strauss [10] gave a very simple proof of the stability of the peakons (1.4) for the CH equation in the H^1 norm. Considering a minimization problem with an appropriate constraint, moreover, Constantin and Molinet [11] proved the same by a variational method.

It is found that the corresponding conservation laws of the DP equation are much weaker than those of the CH equation. In particular, the conservation law $E_2(u)$ of the DP equation is equivalent to $\|u\|_{L^2}^2$, while $F_2(u)$ of the CH equation is $\|u\|_{H^1}^2$. In fact, by Fourier transformation, we have

$$E_2(u) = \int_{\mathbb{R}} yv\mathrm{d}x = \int_{\mathbb{R}} \frac{1+\xi^2}{4+\xi^2}|\hat{u}(\xi)|^2\mathrm{d}\xi \sim \|\hat{u}\|_{L^2}^2 = \|u\|_{L^2}^2,$$

where $\hat{u}$ is the Fourier transform in x of u. Therefore, the stability issue of the peakons for the DP equation is more subtle. By extending the approach in [10] and constructing a Lyapunov function, Lin and Liu [12] proved the stability of the peakons (1.4) for the DP equation.

Recently, there has been considerable interest in the following DP equation with a strong dispersive term [13–17]:

$$u_t - u_{txx} + 4uu_x + \gamma(u - u_{xx})_x = 3u_xu_{xx} + uu_{xxx}, \quad t > 0,\ x \in \mathbb{R}. \tag{1.7}$$

The mathematical questions with which these papers were concerned included well-posedness of the initial value problem, blow-up phenomena, existence of global weak solutions, wave-breaking phenomena, and exact traveling wave solutions. Note that Eq. (1.7) can be obtained from Eq. (1.1) with $\alpha = 1$, $c_0 = \gamma$, $\gamma_0 = -\gamma$, $c_1 = -2$, $c_2 = c_3 = 1$, and it has the Lax pair

$$(1 - \partial_x^2)\psi_x = \mu m\psi, \qquad \psi_t + \frac{1}{\mu}\psi_{xx} + (u - \gamma)\psi_x - u_x\psi = 0,$$

where $m = u - u_{xx}$. Moreover, Eq. (1.7) has the following Hamiltonian structure:

$$m_t = -B_0\frac{\delta H_{-1}}{\delta m} = -B_1\frac{\delta H_0}{\delta m},$$