Nonlinear Analysis 74 (2011) 2765–2777

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

On certain classes of functional inclusions with causal operators in Banach spaces

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ARTICLE INFO

Article history:
Received 12 November 2009
Accepted 23 December 2010

MSC:
primary 47J05
secondary 34A60
34G25
34K09
45D05
47H04
47H08
47H10

Keywords:
Causal operator
Functional inclusion
Cauchy problem
Functional differential inclusion
Volterra integro-differential inclusion
Continuous dependence
Measure of noncompactness
Fixed point
Topological degree
Multivalued map
Condensing map

ABSTRACT

We introduce the notion of a multivalued causal operator and consider an abstract Cauchy problem in a Banach space for various classes of functional inclusions with causal operators. The methods of the topological degree theory for condensing maps are applied to obtain local and global existence results for this problem and to study the continuous dependence of a solution set on initial data. As application we generalize some existence results for semilinear functional differential inclusions and Volterra integro-differential inclusions with delay.

1. Introduction

At the present time the study of systems governed by differential and functional equations with causal operators attracts much attention. The term causal arises from the engineering and the notion of a causal operator turns out to be a powerful tool for unifying problems in ordinary differential equations, integro-differential equations, functional differential equations with finite or infinite delay, Volterra integral equations, neutral functional equations, etc. (see the monograph [1]). Various problems for functional differential equations with causal operators were considered in recent papers [2–5]. In particular, the existence, uniqueness, and continuous dependence of solutions to Cauchy problem in a Banach space are studied in [2,5].

In the present paper we introduce the notion of a multivalued causal operator and consider an abstract Cauchy problem in a Banach space for various classes of functional inclusions with causal operators. The methods of the topological degree

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0362-546X/$ – see front matter
doi:10.1016/j.na.2010.12.024

theory for condensing maps are applied to obtain local and global existence results for this problem and to study the continuous dependence of a solution set on initial data. As application we generalize some existence results for semilinear functional differential inclusions and Volterra integro-differential inclusions with delay.

The paper is organized as follows. In the next section we present necessary information from the theory of condensing multivalued maps. Further, we introduce the notion of a multivalued causal operator and illustrate it by certain examples. In Section 3 we formulate the abstract Cauchy problem for a functional inclusion containing the composition of multivalued and single-valued causal operators. We study the properties of the multioperator whose fixed points induce solutions of the problem. In particular, sufficient conditions under which this multioperator is condensing are presented. On this basis, in Section 4 we give local and global existence results and describe the continuous dependence of the solution set on initial data. In Section 5 the case of inclusions with lower semicontinuous causal multioperators is considered. In the last section, applying abstract results, we generalize some existence results for semilinear functional differential inclusions and Volterra integro-differential inclusions with delay in a Banach space.

2. Preliminaries

2.1. Multimaps and measures of noncompactness

Let X be a metric space, Y a Banach space, and $P(Y)$ denote the collection of all nonempty subsets of Y. We denote

$$C(Y) = \{D \in P(Y) : D \text{ is closed}\};$$
$$Cv(Y) = \{D \in C(Y) : D \text{ is convex}\},$$
$$K(Y) = \{D \in P(Y) : D \text{ is compact}\};$$
$$Kv(Y) = \{D \in K(Y) : D \text{ is convex}\}.$$

We recall some notions (see e.g. [6,7] for further details).

Definition 1. A multivalued map (multimap) $\mathcal{F} : X \to P(Y)$ is

(i) upper semicontinuous (u.s.c.) if $\mathcal{F}^{-1}(\mathcal{V}) = \{x \in X : \mathcal{F}(x) \subset \mathcal{V}\}$ is an open subset of X for every open set $\mathcal{V} \subset Y$;
(ii) lower semicontinuous (l.s.c.) if $\mathcal{F}^{-1}(\mathcal{W})$ is a closed subset of X for every closed set $\mathcal{W} \subset Y$;
(iii) closed, if its graph $G_F = \{(x, y) : x \in X, y \in F(x)\}$ is a closed subset of $X \times Y$.

In what follows we will need the following property.

Proposition 1 (*Theorem 1.1.12 of [7]*)**.** *Let a closed multimap $\mathcal{F} : X \to K(Y)$ be quasicompact, i.e., for each compact set $K \subset X$ the set $\mathcal{F}(K) = \cup_{x \in K} F(x)$ is relatively compact in Y. Then the multimap $\mathcal{F}$ is u.s.c.*

Sometimes we will denote a multimap by the symbol $\mathcal{F} : X \multimap Y$.

Definition 2. A multifunction $\mathcal{F} : [a, b] \subset \mathbb{R} \to K(Y)$ is said to be strongly measurable if there exists a sequence $\mathcal{F}_n : [a, b] \to K(Y)$, $n = 1, 2, \ldots$ of step multifunctions such that

$$\lim_{n\to\infty} h(\mathcal{F}_n(t), \mathcal{F}(t)) = 0 \quad \text{for } \mu - \text{a.e. } t \in [a, b],$$

where μ denotes the Lebesgue measure on $[a, b]$ and h is the Hausdorff metric on $K(Y)$.

Definition 3. Let $\mathcal{E}$ be a Banach space and $(\mathcal{A}, \geq)$ a partially ordered set. A function $\beta : P(\mathcal{E}) \to \mathcal{A}$ is called a measure of noncompactness (MNC) in $\mathcal{E}$ if

$$\beta(\overline{co}\Omega) = \beta(\Omega) \quad \text{for every } \Omega \in P(\mathcal{E}).$$

A MNC β is called

(i) monotone, if $\Omega_0, \Omega_1 \in P(\mathcal{E})$, $\Omega_0 \subseteq \Omega_1$ implies $\beta(\Omega_0) \leq \beta(\Omega_1)$;
(ii) nonsingular, if $\beta(\{e\} \cup \Omega) = \beta(\Omega)$ for every $e \in \mathcal{E}$, $\Omega \in P(\mathcal{E})$;
(iii) invariant with respect to union with compact sets, if $\beta(\{K\} \cup \Omega) = \beta(\Omega)$ for every relatively compact set $K \subset \mathcal{E}$, $\Omega \in P(\mathcal{E})$;
(iv) real, if $\mathcal{A}$ is $[0, +\infty)$ with the natural order. If $\mathcal{A}$ is a cone in a normed space, we say that the MNC β is
(v) algebraically semiadditive if $\beta(\Omega_0 + \Omega_1) \leq \beta(\Omega_0) + \beta(\Omega_1)$ for every $\Omega_0, \Omega_1 \in P(\mathcal{E})$;
(vi) regular, if $\beta(\Omega) = 0$ is equivalent to the relative compactness of Ω.

As an example of MNC satisfying all the above properties we can consider
the Hausdorff MNC

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ has a finite } \varepsilon\text{-net}\}.$$