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Existence of real eigenvalues of real tensors

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ABSTRACT

We use the Brouwer degree to establish the existence of real eigenpairs of higher order real tensors in various settings. Also, we provide some finer criteria for the existence of real eigenpairs of two-dimensional real tensors and give a complete classification of the Brouwer degree zero and ± 2 maps induced by general third order two-dimensional real tensors.

1. Introduction

Let $\mathbb{R}$ be the real field, we consider an m-order n-dimensional tensor $\mathcal{A}$ consisting of n^m entries in $\mathbb{R}$:

$$\mathcal{A} = (a_{i_1 \cdots i_m}), \qquad a_{i_1 \cdots i_m} \in \mathbb{R}, \quad 1 \le i_1, \ldots, i_m \le n.$$

To an n-vector $x = (x_1, \ldots, x_n)$, real or complex, we define an n-vector:

$$\mathcal{A}x^{m-1} := \left(\sum_{i_2,\ldots,i_m=1}^{n} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} \right)_{1 \le i \le n}.$$

The following were first introduced and studied by Qi [1–3] and Lim [4].

Definition 1.1. Let $\mathcal{A}$ be an m-order n-dimensional real tensor. Assume that $\mathcal{A}x^{m-1}$ is not identically zero. We say $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is an eigenpair if they satisfy the equation $\mathcal{A}x^{m-1} = \lambda x^{[m-1]}$, where $x^{[m-1]} = (x_1^{m-1}, \ldots, x_n^{m-1})$. We say it is an H-eigenpair if they are both real.

Definition 1.2. Let $\mathcal{A}$ be an m-order n-dimensional real tensor. Assume that $\mathcal{A}x^{m-1}$ is not identically zero. We say $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is an E-eigenpair if they satisfy the equation $\mathcal{A}x^{m-1} = \lambda x$. We say it is a Z-eigenpair if they are both real.

The above notions of eigenvalues were generalized by [5] as follows.

Definition 1.3. Let $\mathcal{A}$ and $\mathcal{B}$ be two m-order n-dimensional real tensors. Assume that both $\mathcal{A}x^{m-1}$ and $\mathcal{B}x^{m-1}$ are not identically zero. We say $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is an eigenvalue–eigenvector of $\mathcal{A}$ relative to $\mathcal{B}$, if the n-system of

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equations:

$$(\mathcal{A}-\lambda\mathcal{B})x^{m-1}=0,$$
$$\text{i.e.}\ \sum_{i_2,\ldots,i_m=1}^{n}(A_{ii_2\cdots i_m}-\lambda B_{ii_2\cdots i_m})x_{i_2}\cdots x_{i_m}=0,\quad i=1,2,\ldots,n$$

possesses a solution.

Remark 1.4. If $\mathcal{B}=\mathcal{I}$, the unit tensor $\mathcal{I}=(\delta_{i_1\cdots i_m})$, then the $\mathcal{B}$-eigenvalues are the eigenvalues and the real $\mathcal{B}$-eigenpairs are the H-eigenpairs.

For $m=2\ell$ and let I_2 be the $n\times n$ unit matrix. If $\mathcal{B}=I_2^{\ell}$, the tensor product of ℓ copies of the unit matrices I_2, then the $\mathcal{B}$-eigenvalues are the E-eigenvalues, and the real $\mathcal{B}$-eigenpairs are called the Z-eigenpairs [1–3].

Previously, some results regarding the existence of H and Z eigenpairs have been established. We list a few here for reference.

Theorem 1.5 (*Corollary 3.7[5]*)**.** *If $\mathcal{A}$ is symmetric with m even, then $\mathcal{A}$ has at least nH – resp. Z – eigenvalues counting multiplicities with n distinct pairs of H (resp. Z) eigenvectors.*

Theorem 1.6 (*Generalized Perron–Frobenius Weak Version [6]*)**.** *If $\mathcal{A}$ is a nonnegative tensor of order m and dimension n, then there exist $\lambda_0\geq 0$ and a nonnegative vector $x_0\neq 0$ such that $\mathcal{A}x_0^{m-1}=\lambda_0 x_0^{[m-1]}$.*

Theorem 1.7 (*Generalized Perron–Frobenius Strong Version [6]*)**.** *If $\mathcal{A}$ is an irreducible nonnegative tensor of order m and dimension n, then the pair (λ_0,x_0) in the previous theorem satisfies the following conditions:*

1. *The eigenvalue λ_0 is positive.*
2. *The eigenvector x_0 is positive, i.e. all components of x_0 are positive.*
3. *If λ is an eigenvalue with nonnegative eigenvector, then $\lambda=\lambda_0$. Moreover, the nonnegative eigenvector is unique up to a multiplicative constant.*
4. *If λ is an eigenvalue of $\mathcal{A}$, then $|\lambda|\leq\lambda_0$.*

However, if we drop either the symmetry or the nonnegativeness assumption on $\mathcal{A}$, the existence of H-eigenpair has not yet been proven. As for the Z-eigenvalue problem, it was shown in [7], for $m>2$, the degree of the Z-characteristic polynomial of a generic tensor $\mathcal{A}$ is $d_Z=((m-1)^n-1)/(m-2)=(m-1)^{n-1}+(m-1)^{n-2}+\cdots+(m-1)+1$, so it is easily seen that when m is odd or n is odd, d_Z is odd and hence has a Z-eigenpair. However, when dealing with a pair of tensors $\mathcal{A}$ and $\mathcal{B}$ in general, the degree of the generalized characteristic polynomial is difficult to determine. The main purpose of this paper is to use the Brouwer degree to overcome this obstacle.

2. A brief review of the Brouwer degree

In this section, we review and prove some basic facts regarding the Brouwer degree of a map.

Definition 2.1. Let $f:S^n\to S^n$ be a continuous map. The degree of f, denoted $\deg f\in\mathbb{Z}$, is the integer such that $f_*([a])=(\deg f)[a]$ for all $[a]\in\widetilde{H}_n(S^n;\mathbb{Z})\cong\mathbb{Z}$.

Two maps $f,g:S^n\to S^n$ have the same degree if and only if they are homotopic. Furthermore, the constant map has degree zero, the identity map has degree one, and the antipodal map has degree $(-1)^{n+1}$. We now prove the following, which is a generalization of the well-known Brouwer fixed point theorem from algebraic topology.

Lemma 2.2. *Let $f,g:S^{n-1}\to S^{n-1}$ be two continuous maps such that at least one of them is not null-homotopic. If n is odd, then $\exists\, x_0\in S^{n-1}$ such that $f(x_0)=g(x_0)$ or $f(x_0)=-g(x_0)$.*

Proof. We argue contrapositively. Suppose $\forall x\in S^{n-1}, f(x)\pm g(x)\neq 0$. So for all $0\leq t\leq 1$, $tf(x)\pm(1-t)g(x)\neq 0$. Without loss of generality, we may assume that g is not null-homotopic, i.e. $\deg g\neq 0$. It follows, via the homotopy $\frac{tf(x)+(1-t)g(x)}{\|tf(x)+(1-t)g(x)\|}$, that f is homotopic to g, which implies $\deg f=\deg g$. On the other hand, f is homotopic to $-g$ via the homotopy $\frac{tf(x)-(1-t)g(x)}{\|tf(x)-(1-t)g(x)\|}$, which implies $\deg f=(-1)^n\deg g$. Since $\deg g\neq 0$, this can happen only if n is even. □

Using the same argument, we can show the following lemma.

Lemma 2.3. *If $f,g:S^{n-1}\to S^{n-1}$ are two continuous maps such that $\deg f\neq\pm\deg g$, then $\exists\, x_0\in S^{n-1}$ such that $f(x_0)=g(x_0)$ or $f(x_0)=-g(x_0)$.*

Again by the same argument, we have the following.

Theorem 2.4 (*Brouwer*)**.** *If $f:S^{n-1}\to S^{n-1}$ is continuous with $\deg f\neq(-1)^n$, then f must have a fixed point.*