Nonlinear Analysis 74 (2011) 2887–2894

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Pseudo-almost automorphic mild solutions to semilinear integral equations in a Banach space

Zhi-Han Zhao [a], Yong-Kui Chang [a,*], G.M. N'Guérékata [b]

[a] *Department of Mathematics, Lanzhou Jiaotong University, Lanzhou, Gansu 730070, PR China*

[b] *Department of Mathematics, Morgan State University, 1700 E. Cold Spring Lane, Baltimore, MD 21251, USA*

ARTICLE INFO

Article history:
Received 28 December 2010
Accepted 19 January 2011

MSC:
44A35
42A85
42A75

Keywords:
Pseudo-almost automorphic function
Semilinear integral equations
Integral resolvent family
Fixed point

ABSTRACT

In this paper, we consider the existence of pseudo-almost automorphic solutions of the semilinear integral equation $x(t) = \int_{-\infty}^{t} a(t-s)[Ax(s) + f(s, x(s))]\mathrm{d}s$, $t \in \mathbb{R}$ in a Banach space $\mathbb{X}$, where $a \in L^1(\mathbb{R}_+)$, A is the generator of an integral resolvent family of linear bounded operators defined on the Banach space $\mathbb{X}$, and $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is a pseudo-almost automorphic function. The main results are proved by using integral resolvent families combined with the theory of fixed points.

1. Introduction

In this paper, we are mainly concerned with the existence of pseudo-almost automorphic mild solutions to the class of semilinear integral equations in the abstract form

$$x(t) = \int_{-\infty}^{t} a(t-s)[Ax(s) + f(s, x(s))]\mathrm{d}s, \quad t \in \mathbb{R}, \tag{1.1}$$

where $a \in L^1(\mathbb{R}_+)$, $A : D(A) \subseteq \mathbb{X} \to \mathbb{X}$ is the generator of an integral resolvent family defined on a complex Banach space $\mathbb{X}$, and $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is a pseudo-almost automorphic function satisfying some suitable conditions (see Section 3).

The concept of pseudo-almost automorphic functions suggested by N'Guérékata [1, page 40] was developed by Xiao et al. [2]. It is a natural generalization of both the classical almost automorphy in the sense of Bochner [3,4] and that of pseudo-almost periodicity due to Zhang [5–7]. Since then, these functions have generated lot of developments and applications. For more details we refer the reader to [1,8–10] and the references therein. The existence of almost automorphic and pseudo-almost automorphic solutions are among the most attractive topics in the qualitative theory of differential equations because of their significance and applications in physics, mechanics and mathematical biology. In recent years, the existence of automorphic and pseudo-almost automorphic solutions on different kinds of differential equations have been investigated in many publications such as [11–22] and the references therein.

Recently, Cuevas and Lizama [23] studied the existence and uniqueness of almost automorphic solutions for problem (1.1). In [24], the authors investigated the existence and regularity of compact almost automorphic solutions

* Corresponding author.
E-mail addresses: zhaozhihan841110@126.com (Z.-H. Zhao), lzchangyk@163.com (Y.-K. Chang), Gaston.N'Guerekata@morgan.edu (G.M. N'Guérékata).

0362-546X/$ – see front matter
doi:10.1016/j.na.2011.01.018

for problem (1.1). And in [25], we considered the existence and uniqueness of pseudo-almost automorphic and weighted pseudo-almost automorphic mild solutions to a semilinear differential equation in Hilbert spaces. However, to the best of our knowledge, the existence of pseudo-almost automorphic solutions to problem (1.1) is an untreated topic. Motivated by the above-mentioned works [23,24], the main purpose of this paper is to investigate the existence of pseudo-almost automorphic solutions to problem (1.1). Theorem 3.3 and other main results following this extend results in [24] and can be seen as a contribution to this emerging field.

The rest of this paper is organized as follows. In Section 2, we present some basic definitions, lemmas and preliminary facts which will be needed in what follows. In Section 3, we prove the existence and uniqueness (or existence) of pseudo-almost automorphic mild solutions for the semilinear integral equation (1.1).

2. Preliminaries

In this section, we introduce some basic definitions, notations, lemmas and technical results which will be used in what follows. For more details on this section, we refer the reader to [18,19,24].

Throughout this paper, we assume that $(\mathbb{X}, \|\cdot\|)$ is a complex Banach space. The notation $C(\mathbb{R}, \mathbb{X})$ stands for the space of all continuous functions from $\mathbb{R}$ into $\mathbb{X}$. We let $\mathrm{BC}(\mathbb{R}, \mathbb{X})$ (respectively, $\mathrm{BC}(\mathbb{R}\times\mathbb{X}, \mathbb{X})$) be the space of all bounded continuous functions from $\mathbb{R}$ into $\mathbb{X}$ (respectively, the collection of all jointly bounded continuous functions from $\mathbb{R}\times\mathbb{X}$ into $\mathbb{X}$). Note that $\mathrm{BC}(\mathbb{R}, \mathbb{X})$ is a Banach space with the sup norm $\|\cdot\|_\infty$; we write $\mathrm{BC}(\mathbb{R})$ when $\mathbb{X} = \mathbb{R}$. Furthermore, we denote by $\mathfrak{B}(\mathbb{X})$ the space of bounded linear operators from $\mathbb{X}$ into $\mathbb{X}$ endowed with the operator topology, and the notation $\rho(A)$ stands for the resolvent set of A.

We recall that the Laplace transform of a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{X})$ is given by

$$\mathfrak{L}(f)(\lambda) := \hat{f}(\lambda) := \int_0^\infty \mathrm{e}^{-\lambda t} f(t)\mathrm{d}t, \quad \operatorname{Re}\lambda > \omega,$$

where the integral is absolutely convergent for $\operatorname{Re}\lambda > \omega$. In order to establish an operator theoretical approach to Eq. (1.1), we recall the following definition.

Definition 2.1 (*[26]*)**.** Let A be a closed linear operator with domain $D(A) \subseteq \mathbb{X}$. We say that A is the generator of an integral resolvent if there exist $\omega \geq 0$ and a strongly continuous function $S : \mathbb{R}_+ \to \mathfrak{B}(\mathbb{X})$ such that $\left\{\frac{1}{\hat{a}(\lambda)} : \operatorname{Re}\lambda > \omega\right\} \subseteq \rho(A)$ and

$$\left(\frac{1}{\hat{a}(\lambda)} I - A\right)^{-1} x = \int_0^\infty \mathrm{e}^{-\lambda t} S(t) x \mathrm{d}t, \quad \operatorname{Re}\lambda > \omega,\ x \in \mathbb{X}.$$

In this case, $S(t)$ is called the integral resolvent family generated by A.

Now, we establish several relations between the integral resolvent family and its generator. The following result is a direct consequence of [27, Proposition 3.1 and Lemma 2.2].

Lemma 2.1. *Let $S(t)$ be the integral resolvent family on $\mathbb{X}$ with generator A. Then the following properties hold:*

(a) *$S(t)D(A) \subseteq D(A)$ and $AS(t)x = S(t)Ax$ for all $x \in D(A)$ and $t \geq 0$.*

(b) *Let $x \in D(A)$ and $t \geq 0$. Then*

$$S(t)x = a(t)x + \int_0^t a(t-s)AS(s)x\mathrm{d}s.$$

(c) *Let $x \in \mathbb{X}$ and $t \geq 0$. Then $\int_0^t a(t-s)S(s)x\mathrm{d}s \in D(A)$ and*

$$S(t)x = a(t)x + A\int_0^t a(t-s)S(s)x\mathrm{d}s.$$

In particular, $S(0) = a(0)I$.

Remark 2.1. For more on integral resolvent families and related issues, we refer the reader to [27–32].

Definition 2.2 (*[3]*)**.** A continuous function $f : \mathbb{R} \to \mathbb{X}$ is said to be almost automorphic if for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$ there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that

$$g(t) := \lim_{n\to\infty} f(t + s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} g(t - s_n) = f(t)$$

for each $t \in \mathbb{R}$. The collection of all such functions will be denoted by $AA(\mathbb{R}, \mathbb{X})$.