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Existence, blow-up and exponential decay estimates for a nonlinear wave equation with boundary conditions of two-point type

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ABSTRACT

This paper is devoted to studying a nonlinear wave equation with boundary conditions of two-point type. First, we state two local existence theorems and under the suitable conditions, we prove that any weak solutions with negative initial energy will blow up in finite time. Next, we give a sufficient condition to guarantee the global existence and exponential decay of weak solutions. Finally, we present numerical results.

1. Introduction

In this paper, we consider the following nonlinear wave equation with initial conditions and boundary conditions of two-point type

$$u_{tt} - u_{xx} + u + \lambda u_t = |u|^{p-2}u, \quad 0 < x < 1,\ t > 0, \tag{1.1}$$

$$u_x(0,t) = -|u(0,t)|^{\alpha-2}u(0,t) + \lambda_0 u_t(0,t) + \widetilde{h}_1(t)u(1,t) + \widetilde{\lambda}_1 u_t(1,t), \quad t > 0, \tag{1.2}$$

$$-u_x(1,t) = -|u(1,t)|^{\beta-2}u(1,t) + \lambda_1 u_t(1,t) + \widetilde{h}_0(t)u(0,t) + \widetilde{\lambda}_0 u_t(0,t), \quad t > 0, \tag{1.3}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \tag{1.4}$$

where $\lambda_0, \lambda_1, \widetilde{\lambda}_0, \widetilde{\lambda}_1, \lambda, p$ are constants and $u_0, u_1, \widetilde{h}_0, \widetilde{h}_1$ are given functions satisfying conditions specified later.

The wave equation

$$u_{tt} - \Delta u = f(x,t,u,u_t), \tag{1.5}$$

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with the different boundary conditions, has been extensively studied by many authors, see [1–17] and references therein. In these works, many interesting results about the existence, regularity and the asymptotic behavior of solutions were obtained.

In [13], Munoz-Rivera and Andrade dealt with the global existence and exponential decay of solutions of the nonlinear one-dimensional wave equation with a viscoelastic boundary condition.

In [14–16], Santos also studied the asymptotic behavior of solutions to a coupled system of wave equations having integral convolutions as memory terms. The main results show that solutions of that system decay uniformly in time, with rates depending on the rate of decay of the kernel of the convolutions.

In [17], the global existence and regularity of weak solutions for the linear wave equation

$$u_{tt} - u_{xx} + Ku + \lambda u_t = f(x,t), \quad 0 < x < 1,\ t > 0, \tag{1.6}$$

with the initial conditions as in (1.4) and the two-point boundary conditions

$$\begin{cases} u_x(0,t) = h_0 u(0,t) + \lambda_0 u_t(0,t) + \widetilde{h}_1 u(1,t) + \widetilde{\lambda}_1 u_t(1,t) + g_0(t), \\ -u_x(1,t) = h_1 u(1,t) + \lambda_1 u_t(1,t) + \widetilde{h}_0 u(0,t) + \widetilde{\lambda}_0 u_t(0,t) + g_1(t), \end{cases} \tag{1.7}$$

were proved, where $h_0, h_1, \widetilde{h}_0, \widetilde{h}_1, \lambda_0, \lambda_1, \widetilde{\lambda}_0, \widetilde{\lambda}_1, K, \lambda$ are constants and u_0, u_1, g_0, g_1, f are given functions. Furthermore, the exponential decay of solutions were also given there by using the Lyapunov method.

We note more that, the following nonhomogeneous boundary conditions were considered by Hellwig [18, p. 151]:

$$\begin{cases} \alpha_{01}u(0,t) + \alpha_{02}u_x(0,t) + \alpha_{03}u_t(0,t) + \beta_{01}u(1,t) + \beta_{02}u_x(1,t) + \beta_{03}u_t(1,t) = f_0(t), \\ \alpha_{11}u(0,t) + \alpha_{12}u_x(0,t) + \alpha_{13}u_t(0,t) + \beta_{11}u(1,t) + \beta_{12}u_x(1,t) + \beta_{13}u_t(1,t) = f_1(t), \end{cases} \tag{1.8}$$

where $\alpha_{ij}, \beta_{ij},\ i = 0, 1,\ j = 1, 2, 3$ are constants and $f_0(t), f_1(t)$ are given functions.

Let $\Delta = \alpha_{02}\beta_{12} - \alpha_{12}\beta_{02} \neq 0$, (1.8) is transformed into

$$\begin{cases} u_x(0,t) = h_0 u(0,t) + \lambda_0 u_t(0,t) + \widetilde{h}_1 u(1,t) + \widetilde{\lambda}_1 u_t(1,t) + g_0(t), \\ -u_x(1,t) = h_1 u(1,t) + \lambda_1 u_t(1,t) + \widetilde{h}_0 u(0,t) + \widetilde{\lambda}_0 u_t(0,t) + g_1(t), \end{cases} \tag{1.9}$$

in which

$$\begin{cases} h_0 = \frac{1}{\Delta}(\beta_{02}\alpha_{11} - \beta_{12}\alpha_{01}), & h_1 = \frac{1}{\Delta}(\alpha_{02}\beta_{11} - \alpha_{12}\beta_{01}), \\ \lambda_0 = \frac{1}{\Delta}(\beta_{02}\alpha_{13} - \beta_{12}\alpha_{03}), & \lambda_1 = \frac{1}{\Delta}(\alpha_{02}\beta_{13} - \alpha_{12}\beta_{03}), \\ \widetilde{h}_0 = \frac{1}{\Delta}(\alpha_{02}\alpha_{11} - \alpha_{12}\alpha_{01}), & \widetilde{h}_1 = \frac{1}{\Delta}(\beta_{02}\beta_{11} - \beta_{12}\beta_{01}), \\ \widetilde{\lambda}_0 = \frac{1}{\Delta}(\alpha_{02}\alpha_{13} - \alpha_{12}\alpha_{03}), & \widetilde{\lambda}_1 = \frac{1}{\Delta}(\beta_{02}\beta_{13} - \beta_{12}\beta_{03}), \\ g_0(t) = \frac{1}{\Delta}(\beta_{12}f_0(t) - \beta_{02}f_1(t)), & g_1(t) = \frac{1}{\Delta}(\alpha_{12}f_0(t) - \alpha_{02}f_1(t)). \end{cases} \tag{1.10}$$

The main goal of this paper is to extend some results of [17]. Motivated by the problem of the exponential decay of solutions for (1.6)–(1.7), we establish a blow up result and a decay result for the general problem (1.1)–(1.4).

In Theorem 3.1, by applying techniques as in [11] with some necessary modifications and with some restrictions on the initial data, we prove that the solution of (1.1)–(1.4) blows up in finite time.

In Theorem 4.1, by the construction of a suitable Lyapunov functional we also prove that the solution will exponential decay if the initial energy is positive and small.

The paper consists of five sections. In Section 2, we present some preliminaries and the existence results. The proofs of Theorems 3.1 and 4.1 are done in Sections 3 and 4. Finally, in Section 5 we give numerical results.

2. Existence and uniqueness of solution

First, we put $\Omega = (0, 1)$; $Q_T = \Omega \times (0, T),\ T > 0$ and we denote the usual function spaces used in this paper by the notations $C^m(\overline{\Omega}), W^{m,p} = W^{m,p}(\Omega), L^p = W^{0,p}(\Omega), H^m = W^{m,2}(\Omega), 1 \le p \le \infty,\ m = 0, 1, \ldots$ Let $\langle \cdot, \cdot \rangle$ be either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space. The notation $\|\cdot\|$ stands for the norm in L^2 and we denote by $\|\cdot\|_X$ the norm in the Banach space X. We call X' the dual space of X. We denote by $L^p(0, T; X)$, $1 \le p \le \infty$ for the Banach space of the real functions $u : (0, T) \to X$ measurable, such that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p \, dt \right)^{1/p} < \infty \quad \text{for } 1 \le p < \infty,$$