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# The relaxation-time limit in the compressible Euler–Maxwell equations

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ABSTRACT

The aim of this paper is to study multidimensional Euler–Maxwell equations for plasmas with short momentum relaxation time. The convergence for the smooth solutions to the compressible Euler–Maxwell equations toward the solutions to the smooth solutions to the drift–diffusion equations is proved by means of the Maxwell iteration, as the relaxation time tends to zero. Meanwhile, the formal derivation of the latter from the former is justified.



## 1. Introduction

The Euler–Maxwell systems and the drift–diffusion models are well known as describing the behavior of electric flow in semiconductors or plasma. Hence a study on relations between these models is an important topic not only in mathematics but also in engineering. The relation can be formally obtained by letting the relaxation time tend to zero. Actually, the drift–diffusion model is derived from the Euler–Maxwell model by this limit regime. This limit regime is called a relaxation time limit.

This paper is devoted to the relaxation time limit problem for the (rescaled) isentropic Euler–Maxwell systems which takes the following form [1–3]

$$\partial_t n + \frac{1}{\tau}\mathrm{div}\,(nu) = 0, \tag{1.1}$$

$$\partial_t (nu) + \frac{1}{\tau}\mathrm{div}\,(nu\otimes u) + \frac{1}{\tau}\nabla p(n) + \frac{nu}{\tau^2} = -\frac{n}{\tau}(E + u\times B), \tag{1.2}$$

$$\partial_t E - \frac{1}{\tau}\nabla\times B = \frac{nu}{\tau}, \quad \mathrm{div}\,E = b(x) - n, \tag{1.3}$$

$$\partial_t B + \frac{1}{\tau}\nabla\times E = 0, \quad \mathrm{div}\,B = 0, \tag{1.4}$$

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where $n, u, E$ and $B$ denote the scaled macroscopic density, the mean velocity vector, the scaled electric field and the magnetic field, respectively. They are functions of a three-dimensional position vector $x \in \mathbb{T}^3$ and the time $t > 0$, where $\mathbb{T}^3 = \left(\frac{\mathbb{R}}{2\pi\mathbb{Z}}\right)^3$ is the three-dimensional torus. $p = p(n)$ is a given strictly increasing function and denotes the pressure. $b(x)$ stands for the prescribed density of positive charged background ions (doping profile). The dimensionless parameter $\tau$ stands for the momentum relaxation time. div, $\nabla$, $\Delta$ and $\otimes$ are the respective $x$-divergence operator, the gradient operator, the Laplacian operator, and the symbol for the tensor products of two vectors. Note that the scaling

$$t = \tau t'$$

converts (1.1)–(1.4) back into the original one-fluid model in [1–3] with $t'$ as its time variable. The scaled-time variable $t$ was first introduced in [4] by Marcati and Natalini to study the relation between the isentropic or isothermal hydrodynamical and drift–diffusion models. Since then, this kind of limit problem has been investigated by various authors for entropy weak solutions [5–8] and for smooth solutions [9–12].

Applying the Maxwell iteration to the momentum equation in (1.2) gives

$$\begin{aligned} nu &= -\tau\nabla p(n) - \tau n(E + u \times B) - \tau\, \mathrm{div}\,(nu \otimes u) - \tau^2 \partial_t(nu) \\ &= -\tau\nabla p(n) - \tau nE + \tau\left(-\tau\nabla p(n) - \tau n(E + u \times B) - \tau\, \mathrm{div}\,(nu \otimes u) - \tau^2\partial_t(nu)\right) \times B \\ &\quad - \tau \mathrm{div}\, \left(\left(-\tau\nabla p(n) - \tau n(E + u \times B) - \tau \mathrm{div}\,(nu \otimes u) - \tau^2\partial_t(nu)\right) \otimes u\right) - \tau^2\partial_t(nu) \\ &= -\tau\nabla p(n) - \tau nE + O(\tau^2). \end{aligned} \tag{1.5}$$

Substituting the truncation $nu = -\tau\nabla p(n) + \tau nE$ into the mass equation (1.1) and Maxwell equations (1.3)–(1.4) and letting $\tau \to 0$, we arrive at the one-fluid drift–diffusion model

$$\begin{cases} \partial_t n = \Delta p(n) - \mathrm{div}\,(nE), \\ \nabla \times E = 0, \quad \mathrm{div}\, E = b(x) - n \end{cases} \tag{1.6}$$

and the curl–div equation as follows

$$\nabla \times B = 0, \quad \mathrm{div}\, B = 0. \tag{1.7}$$

This is because the equation $\nabla \times E = 0$ implies that the electric field is the gradient of some potential function, i.e. $E = \nabla\phi$. From equation $\mathrm{div}\, E = b(x) - n$ we can obtain $E = \nabla\Delta^{-1}(b(x) - n)$. Here, the operator $\nabla\Delta^{-1}$ is the mapping from $L(\mathbb{T}^3)$ into $L(\mathbb{T}^3)$ [13]. Thus Eq. (1.6) is a parabolic–elliptic system, since $p(n)$ is strictly increasing. For the curl–div equations (1.7), we can take $B = 0$ in the class $\mathbf{m}(B) = \int_{\mathbb{T}^3} B dx = 0$.

The goal of this paper is to justify the above formal derivation of the drift–diffusion model for periodic IVPs (initial-value problems) with an emphasis on three dimensional space. To the best of authors' knowledge, there is no result on the relaxation time limit of the compressible Euler–Maxwell model (1.1)–(1.4).

The Euler–Maxwell equations are more intricate than the Euler–Poisson equations, because of the complicated coupling of the Lorentz force. So there have been few studies on the Euler–Maxwell equations and their asymptotic analysis than on the Euler–Poisson equations. See [14–21] and the references therein. The first mathematical study of the Euler–Maxwell equations with extra relaxation terms is due to Chen et al. [22], where a global existence result to weak solutions in the one-dimensional case is established by the fractional step Godunov scheme together with a compensated compactness argument. In 2003, J.W. Jerome established a local smooth solution theory for the Cauchy problem of compressible Hydrodynamic–Maxwell systems (Ref. [23]) via a modification of the classical semigroup-resolvent approach of Kato. Recently, the convergence of the one-fluid (isentropic) Euler–Maxwell system to the compressible Euler–Poisson system is proven by Peng and Wang in [24] via the non-relativistic limit. Peng and Wang also prove that the combined non-relativistic and quasi-neutral limit is the (isentropic) incompressible Euler equations in a uniform background of nonmoving ions with fixed unit density (see [25]). Furthermore, Peng and Wang derive incompressible e-MHD equations from compressible Euler–Maxwell equations via the quasi-neutral regime (see [26]). Yang and Wang [27,28] study the asymptotic limit of two-fluid Euler–Maxwell systems and non-isentropic Euler–Maxwell systems via the non-relativistic regime.

In this paper, we revise the approach in [11,29] to study the relaxation limit problem of Euler–Maxwell equations. Precisely, we assume that the drift–diffusion model (1.6) and the curl–div equations (1.7) have a smooth solution $(n, E, B)$ with initial data $(n(x, 0), E(x, 0), B(x, 0)) = (n_0(x), E_0(x), B_0(x))$. Inspired by the Maxwell iteration above, we construct a formal approximation

$$n_\tau = n, \qquad u_\tau = \tau E - \frac{\nabla P(n)}{n}, \qquad E_\tau = E, \qquad B_\tau = 0 \tag{1.8}$$

for the solution $(n^\tau, u^\tau, E^\tau, B^\tau)$ of (1.1)–(1.4) with initial data

$$n(x, 0) = n_0(x), \qquad u(x, 0) = \tau E_0(x) - \frac{\nabla P(n_0)}{n_0}, \tag{1.9}$$

where $E_0$ satisfies the following compatibility condition

$$\mathrm{div}\, E_0 = b(x) - n_0. \tag{1.10}$$